Asymptotic Methods for Stochastic Volatility Option Pricing: An Explanatory Study

by

Lichen Chen

A Professional Master's Project Report

Submitted to the Faculty

of the

WORCESTER POLYTECHNIC INSTITUTE

In partial fulfillment of the requirements for the

Degree of Master of Science

in

Financial Mathematics

by

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January 2011

APPROVED:

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Professor Hasanjan Sayit, Advisor

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Professor Bogdan Vernescu, Head of Department

Abstract

In this project, we study an asymptotic expansion method for solving stochastic volatility European option pricing problems. We explain the backgrounds and details associated with the method. Specifically, we present in full detail the arguments behind the derivation of the pricing PDEs and detailed calculation in deriving asymptotic option pricing formulas using our own model specifications. Finally, we discuss potential difficulties and problems in the implementation of the methods.

Acknowledgements

I would like to thank my advisor, Professor Hasanjan Sayit, for his guidance and encouragement throughout this process.

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Chapter 1

Introduction

In the traditional Black-Scholes-Merton model for European option pricing, the volatility parameter is assumed to be constant. However, mounting evidence shows that there is a significant discrepancy between Black-Scholes option prices and option prices observed from the market if options of different strikes and maturities on the same stock are priced with the same constant volatility. In order to correct this problem, stochastic volatility models had been proposed which give rise to a lot of new problems in model specification, solving and testing.

In this project, we studied an asymptotic method for solving stochastic volatility option pricing models. The method was first proposed by Fouque, et al. in [8] to solve a stochastic volatility model in which the volatility driving process is an Ornstein-Uhlenbeck (OU) process. Further developments of the method were made to solve models in which volatility was driven by two stochastic processes. We illustrative with full detail how to use those methods to obtain asymptotic option price formulas for given stochastic volatility models.

Although asymptotic option pricing formulas were derived and theoretical error bounds were established, there is little knowledge about how the resulting formulas fit real data. In the comprehensive survey paper by Bakshi, et al. [1], the authors proposed a framework for testing alternative option pricing models. They tested models from three major perspectives: internal consistency of implied parameters with relevant time-series data; out-of-sample pricing; and hedging. To explore the first point, we illustrate with model specifications which can be potentilly used to infer volatility and market risk of volatity parameters in the models.

Chapter 2

Review of the

Black-Scholes-Merton Option

Pricing Theory

The theory of option pricing originated from the seminal works of Black and Scholes [6] and Merton [15], in which they studied the problem of how to assign a fair price to a European option in the sense of No Arbitrage. An arbitrage is defined to be a trading strategy which begins with zero capital and at a later time has positive capital with positive probability without having any risk of loss.

A European call option is a contract that gives its holder the right, but not the obligation, to buy one unit of an underlying asset for a predetermined strike price K on the maturity date T. If S_T is the price of the underlying asset at maturity time T, then value of the this contract at maturity, which is its payoff, equals

$$h(S_T) = (S_T - K)^+. \tag{2.1}$$

In the heart and sole of Black and Scholes' theory is the idea of dynamic hedging: the value of an option can be replicated by a dynamically adjusted portfolio consisting the underlying asset and a position in a money market account. Based on the No Arbitrage assumption, one must have the price of the option equal the price of the replicating portfolio at any time before the maturity of the option. Black and Scholes' theory helped people to understand the nature of an option contract, gave an satisfatory formula for finding the fair price of the option, and shed light on how the writer of the option can hedge his short position.

Between 1979 and 1983, Harrison, Kreps, and Pliska used the general theory of continuous-time stochastic processes to put the Black-Scholes option-pricing theorey on a solid theoretical foundation. Those works include [10, Harrison and Kreps, 1979], [11, Harrison and Pliska, 1981], and [12, Harrison and Pliska, 1983]. Their results enable people to price many other derivative securities and to build option pricing models with considerable degrees of freedom.

In this chapter, we derive the Black-Scholes-Merton formula using the risk-neutral method. The ideas and technical tools used here serve as the foundation of our presentations in the following chapters. Our presentation follows [18] closely.

2.1   Replicating Portfolio

To facilitate our presentation, we first give the mathematical definition of Arbitrage.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $W_t, t \geq 0$, be a Brownian motion, and $\mathcal{F}_t, t \geq 0$ be a filtration associated with the Brownian motion.

Definition 1 (Arbitrage). *An arbitrage is a portfolio value process X_t satisfying $X_0 = 0$ and also satisfying for some time $T > 0$,*

$$\mathbb{P}(X_T \geq 0) = 1,$$

$$\mathbb{P}(X_T > 0) > 0.$$

If there exists a money market account with interest rate r, then an arbitrage can be equivalently defined as a portfolio value process X_t satisfying $X_0 = x_0$ and at a later time $T > 0$

$$\mathbb{P}(X_T \geq e^{rT} x_0) = 1,$$

$$\mathbb{P}(X_T > e^{rT} x_0) > 0.$$

Consider a European call option with maturity T and strike K written on a stock whose price dynamics is modeled by the geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{2.2}$$

in which μ and σ are constant parameters called the *drift* and the *volatility* of the geometric Brownian motion, respectively.

Black and Scholes argued that the value of this option can be replicated by a dynamically adjusted portfolio investing in a money market account with interest rate r and the underlying stock S. Denote the value of this portfolio by X_t and the shares of stocks held by Δ_t. The rest of the money is invested in a money market

account with interest rate r. (A negative value in the account meaning borrowing at the rate of r.) The dynamics of the value of this porfolio is given by

$$dX_t = \Delta_t dS_t + r(X_t - \Delta_t S_t)dt$$

$$= rX_t dt + \Delta_t(\mu - r)S_t dt + \Delta_t \sigma S_t dW_t.$$

(2.3)

And the dynamics of the dicounted portfolio value is given by

$$d(e^{-rt}X_t) = \Delta_t(\mu - r)e^{-rt}dS_t + \Delta_t \sigma e^{-rt}S_t dW_t.$$

(2.4)

Black and Scholes further argued that the value of the option, C, should be a function of time t and the value of the underlying stock S_t. Thus the dynamics of the value of the option can be written as

$$dC(t, S_t) = C_t(t, S_t)dt + C_x(t, S_t)dS_t + \frac{1}{2}C_{xx}(t, S_t)dS_t dS_t$$

$$= \left[C_t(t, S_t) + \mu S_t C_x(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 C_{xx}(t, S_t) \right] dt + \sigma S_t C_x(t, S_t)dW_t.$$

(2.5)

And the dynamics of the discounted option value is

$$d(e^{-rt}C(t, S_t)) = e^{-rt}\left[rC(t, S_t + C_t(t, S_t) + \mu S_t C_x(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 C_{xx}(t, S_t)) \right] dt$$

$$+ e^{-rt}\sigma S_t C_x(t, S_t)dW_t.$$

(2.6)

By replication, we must have

$$d\left(e^{-rt}X_t\right) = d\left(e^{-rt}C(t, S_t)\right).$$

(2.7)

So that we equate (2.4) and (2.6) and have

$$\Delta_t(\mu - r)dS_t + \Delta_t\sigma S_t dW_t$$
$$= \left[rC(t, S_t + C_t(t, S_t) + \mu S_t C_x(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 C_{xx}(t, S_t))\right]dt + \sigma S_t C_x(t, S_t)dW_t.$$

(2.8)

Equating the dW_t terms on both sides of (2.8) gives

$$\Delta_t = C_x(t, S_t).$$

(2.9)

This equation is called *delta-hedging*. It means that at each time t prior to expiration, the number of shares of stocks contained in the hegding portfolio should equal the partial derivative with respect to the stock price of the option value function at that time. The quantity $C_x(t, S_t)$ is called the *delta* of the option. Then we equate the dt terms in (2.9) and have

$$(\mu - r)S_t C_x(t, S_t)$$
$$= -rC(t, S_t) + C_t(t, S_t) + \mu S_t C_x(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 C_x x(t, S_t)$$

(2.10)

for all $t \in [0, T)$.

Simpify (2.10) a little bit, we have

$$C_x(t, x) + rxC_x(t, x) + \frac{1}{2}\sigma^2 x^2 C_{xx}(t, x) = rC(t, x) \qquad (2.11)$$

for all $t \in [0, T)$ and $x \geq 0$. This equation, together with the terminal condition

$$C(T, x) = (x - K)^+, \qquad (2.12)$$

is called the *Black-Scholes-Merton* equation, whose solution gives the No-Arbitrage price of the European call option.

2.2   No Arbitrage Pricing

Notice that in the above Black-Scholes model, interest rate and volatiliity are assumed to be constant. The only source of randomness is the Brownian motion W_t. It is because of this reason that the option can be hedged using the underlying stock and the money market account. However, when we work with stochastic volatility models in which new sources of randomness other than the one driving the stock price are introduced, the above hegding strategy no longer work.

Although option pricing is fully justified when it is accomplished by a hedge for a short position in the derivative security, we are maninly interested in finding a *fair* price of the option in the sense of No Arbitrage. To this end, we derive the Black-Scohles-Merton equation again using the risk-neutral pricing approach. The risk-neutral approach explores the fact that there is a probability measure $\tilde{\mathbb{P}}$, which is equivalent to the probability measure $\mathbb{P}$, under which the discounted stock price process is a martingale. By *equivalent* we mean that the two probability measures

$\mathbb{P}$ and $\tilde{\mathbb{P}}$ agree which sets in $\mathcal{F}$ have probability zero.

Because of this, the discounted price of the option at time t can be written as the expectation of the option's payoff under the risk-neutral measure conditioned on the current information available. It further explores the Markov property of the stock price process, which enables us to write the conditional expectation as a function of time t and the stock price at t. Using the *Feynman-Kac* formula, a partial differential equation can be obtained, whose solution gives the price function of the option that will not lead to Arbitrage opportunities.

The existence of the equivalent probability measure $\tilde{\mathbb{P}}$ is guaranteed by the following *Girsanov*'s theorm.

Theorem 1 (Girsanov). *Let W_t, $0 \leq t \leq T$, be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{F}_t$, $0 \leq t \leq T$, be a filtration for this Brownian motion. Let Θ_t, $0 \leq t \leq T$, ba an adapted process. Define*

$$Z_t = exp\left\{ -\int_0^t \Theta_u dW_u - \frac{1}{2}\int_0^t \Theta_u^2 du \right\}, \tag{2.13}$$

$$\tilde{W}_t = W_t + \int_0^t \Theta_u du, \tag{2.14}$$

and assume that

$$\mathbb{E}\int_0^T \Theta_u^2 Z_u^2 du < \infty. \tag{2.15}$$

Set $Z = Z_T$. Then $\mathbb{E}Z = 1$ and under the probability measure $\tilde{\mathbb{P}}$ given by

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega) dP(\omega), \tag{2.16}$$

for all $A \in \mathcal{F}$, the process $\tilde{W}_t$, $0 \le t \le T$, is a Brownian motion.

Recall that our stock price process is modeled as a geometric Brownian motion under $\mathbb{P}$, whose dynamics is given by (2.2). It is well-known that the solution to (2.2) is

$$S_t = S_0 exp\left\{(\mu - \frac{1}{2}\sigma^2)dt + \sigma dW_t\right\}, \tag{2.17}$$

in which S_0 is the initial value of the process S_t. Thus the discounted stock price process can be written as

$$e^{-rt}S_t = S_0 exp\left\{(\mu - r - \frac{1}{2}\sigma^2)dt + \sigma dW_t\right\}, \tag{2.18}$$

and its differential is

$$d\left(e^{-rt}S_t\right) = (\mu - r)e^{-rt}S_t dt + \sigma e^{-rt}S_t dW_t. \tag{2.19}$$

We rewrite (2.19) as

$$d\left(e^{-rt}S_t\right) = \sigma e^{-rt}S_t\left[\Theta_t dt + dW_t\right], \tag{2.20}$$

in which

$$\Theta_t = \frac{\mu - r}{\sigma}. \tag{2.21}$$

Θ_t is called the market price of risk. It means the excess return over the risk free rate one can expect if one is willing to take one more unit of risk.

Now we introduce the probability measure $\tilde{\mathbb{P}}$ defined in Girsanov's theorem, which uses the market price of risk given by (2.21). In terms of Brownian motion $\tilde{W}_t$ of that theorem, we may write

$$d\left(e^{-rt}S_t\right) = \sigma e^{-rt}S_t d\tilde{W}_t. \tag{2.22}$$

We call $\tilde{\mathbb{P}}$, the measure defined in Girsanov's theorem, the *risk-neutral* measure because it is equivalent to the original measure $\mathbb{P}$ and it renders the discounted stock price $e^{-rt}S_t$ into a martingale.

The undiscounted stock price process S_t has mean rate of return equal to the interest rate under $\tilde{\mathbb{P}}$. This can be seen by replacing $dW_t = -\Theta_t dt + d\tilde{W}_t$ into (2.22). With this substitution, we have

$$dS_t = rS_t dt + \sigma S_t d\tilde{W}_t. \tag{2.23}$$

More generally, we have the following definition for risk-neutral measure:

Definition 2 (Risk-neutral measure)**.** *A probability measure $\tilde{\mathbb{P}}$ is said to be risk-neutral if*

(1)$\tilde{\mathbb{P}}$ and $\mathbb{P}$ are equivalent, and

(2)under $\tilde{\mathbb{P}}$, the discounted stock price $e^{-rt}S_t$ is a martingale.

The following theorem, called the *First Fundamental Theorem of Asset Pricing*, tells us how to check whether an option pricing model is Arbitrage-free:

Theorem 2 (First fundamental theorem of asset pricing)**.** *If a market model admits a risk-neutral probability measure, then it does not admit arbitrage.*

Now let X_t be a replicating portfolio of the European option we are pricing. By replication we have

$$X_T = (S_T - K)^+ a.s. \tag{2.24}$$

Since X_t is always a linear combination of the underlying security and the money market account and the discounted value process of both these two assets are martingales under $\tilde{\mathbb{P}}$, we have that the discounted value of X_t is also a martingale under $\tilde{\mathbb{P}}$. So that

$$e^{-rt}X_t = \tilde{\mathbb{E}}[e^{-rT}X_T|\mathcal{F}_t] = \tilde{\mathbb{E}}[e^{-rT}(S_T - K)^+|\mathcal{F}_t]. \tag{2.25}$$

The value X_t of the replicating portfolio is actually the capital needed at time t in order to construct a hedge of the short position in the derivative security. Hence, we call X_t the price C_t of the derivative security at time t, and (2.25) becomes

$$e^{-rt}C_t = \tilde{\mathbb{E}}[e^{-rT}(S_T - K)^+|\mathcal{F}_t], \tag{2.26}$$

Note that (2.26) can also be written as

$$C_t = \tilde{\mathbb{E}}[e^{-r(T-t)}(S_T - K)^+|\mathcal{F}_t], \tag{2.27}$$

for $0 \leq t \leq T$.

This is called the *risk-neutral pricing formula* for the derivative security.

Notice that we had assumed that there exists a portfolio X_t which replicates the value of the derivative security. However, the exact replicating strategy was not given. But we at least know that such priced derivative security will not lead to Arbitrage opportunities. Thus it is a possible price for the derivative security. Th issue of how to hedge a short position in such a contract may lead to another realm of research. Here we focus on the problem of how to find a No Arbitrage price for the contract.

In cases where we can construct a hedge for a short position in the contract, we say that we have a *complete market* model. Otherwise the model is incomplete. In the Black-Scholes model, we have a complete market. But in the cases of stochastic volatility models, we will have incomplete market.

The following *Second Fundamental Theorem of Asset Pricing* links model completemess with the uniqueness of risk-neutral measure:

Theorem 3 (Second fundamental theorem of asset pricing). *Consider a model that has a risk-neutral probability measure. The model is complete if and only if the risk-neutral measure is unique.*

In the Black-Scholes model, the risk-neutral formula (2.25) can be evaluated explicitly. But if we do not have an explicit formula, we could compute the expectation numerically by beginning at X_t and simulating the paths of X_u for $t \leq u \leq T$. This is the Monte-Carlo simulation method. However, this method could be computationally heavy and will only give the price of the derivative security at time T.

There is another approach to find the value of the conditional expectation. This approach explores the fact that solutions to stochastic differential equations of the form

$$dX_t = \alpha(t, X_t)dt + \beta(t, X_t)dW_t, \tag{2.28}$$

which include geometric Brownian motions as special cases, are Markov processes. And so are measurable functions of Markov processes. The exact definition of a Markov process is given as following:

Definition 3 (Markov process). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let T be a*

fixed positive number, and let $\mathcal{F}_t$, $0 \le t \le T$, be a filtration of sub σ-algebras of $\mathcal{F}$. Consider an adapted stochastic process X_t, $0 \le t \le T$. Assume that for all $0 \le s \le t \le T$ and for every nonnegative, Borel-measurable function f, there is another Borel-measurable function g such that

$$\mathbb{E}[f(X_t)|\mathcal{F}_s] = g(X_s).$$

Then we say that X is a Markov process.

In the above definition, the function f and g are allowed to depend on time. So that we may also write $f = f(t,x)$ and $g = g(t,x)$. Using this fact, and dentoe the stock price at time t by s, we have that there exists a Borel-measurable function $C = C(t,x)$ such that

$$C(t,x) = \mathbb{E}[e^{-r(T-t)}(S_T - K)^+|\mathcal{F}_t] \triangleq \mathbb{E}^{t,x}[e^{-r(T-t)}(S_T - K)^+], \tag{2.29}$$

in which x is the stock price at time t.

Our next important observation is that $e^{-rt}C(t, S_t)$ is a martingale under $\tilde{\mathbb{P}}$. This can be seen as following:

Let $0 \le u \le t \le T$. Since we have

$$e^{-ru}C(u, S_u) = \tilde{\mathbb{E}}[e^{-rT}(S_T - K)^+|\mathcal{F}_u],$$

$$e^{-rt}C(t, S_t) = \tilde{\mathbb{E}}[e^{-rT}(S_T - K)^+|\mathcal{F}_t]$$

from (2.29), we take conditional expectation of the second equation and have

$$\tilde{\mathbb{E}}[e^{-rt}C(t, S_t)|\mathcal{F}_u] = \tilde{\mathbb{E}}[\tilde{\mathbb{E}}[e^{-rt}(S_t - K)^+|\mathcal{F}_t]|\mathcal{F}_u]$$

$$= \tilde{\mathbb{E}}[e^{-ru}(S_u - K)^+|\mathcal{F}_u] \qquad (2.30)$$

$$= e^{-ru}C(u, S_u).$$

Since $e^{-rt}C(t, S_t)$ is a martingale, the dt term in the differential $d(e^{-rt}C(t, S_t))$ must be zero.

$$d(e^{-rt}C(t, S_t)) = e^{-rt}\left[-rCdt + C_t dt + C_x dS + \frac{1}{2}C_{xx}dSdS\right]$$

$$= e^{-rt}\left[-rC + C_t + rC_x + \frac{1}{2}\sigma^2 C_{xx}\right]dt + e^{-rt}\sigma C_x d\tilde{W}_t. \qquad (2.31)$$

Setting the dt term equal to zero, we obtain

$$C_t + rC_x + \frac{1}{2}\sigma^2 C_{xx} - rC = 0.$$

This equation, together with the terminal condition

$$C(T, x) = (x - K)^+,$$

is exactly the Black-Scholes equation for Europrean call option.

The above arguments, which links the conditional expectation to a partial differential equation, can be summarized by the following *Feynman-Kac* theorem:

Theorem 4 (Feynman-Kac). *Consider the stochastic differential equation*

$$dX_u = \beta(u, X_u)du + \gamma(u, X_u)dW_u.$$

Let $h(y)$ be a Borel-measurable function. Fix $T > 0$, and let $t \in [0, T]$ be given.

Define the function

$$g(t, x) = \mathbb{E}^{t,x} h(X_T).$$

(Assume that $\mathbb{E}^{t,x} |h(X_T)| < \infty$.) Then $g(t, x)$ satisfies the partial differential equation

$$g_t(t, x) + \beta(t, x) g_x(t, x) + \frac{1}{2} \gamma^2(t, x) g_{xx}(t, x) = 0$$

and the terminal condition

$$g(T, x) = h(x)$$

for all x.

2.3   Shortfalls of the Black-Scholes-Merton Model

An important concept in the practical use of the Black-Scholes-Merton formula is
implied volatility. The implied volatility of an option is the volatility value that will
equate the BSM formula to the observed option price. In the original BSM model,
volatility is assumed to be constant. If that is the case, then implied volatility should
also be constant However, during the years it has been observed that the volatility
surface of traded options' implied volatilities in terms of time of time-to-maturity
and strike prices exhibit 'smile'-shaped curves, which is called *volatility smile*. This
implies that the constant volatility assumption is highly unrealistic.

One of the major efforts to correct this problem is stochastic volatility models. Major
works along this line include [14], [17], [19], [13], [2], and [16]. Most notably, Renault
and Touzi [16] showed that stochastic volatility models are able to recreate the smile

curves in cases where the volatility and asset driving processes are uncorrelated and the risk premium process is a function of the volatility driving process only. This becomes one of the biggest assets of stochastic volatility models.

Chapter 3

Common Volatility Driving Processes and Their Properties

In this chapter we review two of the most important and widely used volatility driving processes, namely the Ornstein-Uhlenbeck (OU) process and the Cox-Ingersoll-Ross (CIR) process. We give their basic properties and introduce some important concepts associated with them, which will play key roles in the asymptotic expansion methods introduced in chapter 4.

3.1   The Ornstein-Uhlenbeck Process

The Ornstein-Uhlenbeck process is defined as the solution to the stochastic differential equation

$$dY_t = \alpha(m - Y_t)dt + \beta dW_t \tag{3.1}$$

where W_t is a standard Brownian motion. The solution to this stochastic differential equation can be written as

$$Y_t = m + (y - m)e^{-\alpha t} + \beta \int_0^t e^{-\alpha(t-s)} dW_s, \tag{3.2}$$

assuming that the initial value of the process is y. From knowledge about stochatic integrals we have that Y_t is a Gaussian process. The contidional distrinution of Y_t given $Y_0 = y$ is normal with mean $m + (y - m)e^{-\alpha t}$ and standard deviation $\frac{\beta^2}{2\alpha}(1 - e^{-2\alpha t})$. When $t \to \infty$, the mean value and standard deviation of Y_t converge exponentially fast to m and $\frac{\beta^2}{2\alpha}$, respectively. The limit distribution when $t \to \infty$ is called the invariant distribution, or unconditional distribution, of Y_t. More precisely, the invariant distribution Y_0 of a process Y_t is an initial distribution such that for any $t > 0$, Y_t has the same distribution. It is called 'invariant' because it does not change in time. The invariant distribution Y_0 can be found by solving the following differential equation:

$$\frac{d}{dt}\mathbb{E}\{g(Y_t)\} = \frac{d}{dt}\mathbb{E}\{\mathbb{E}\{g(Y_t)|Y_0\}\} = 0,$$

where g is arbitrary.

Since the OU process converges to invariant distribution as time goes on, it is called an asymptotically stationary process, with a Gaussian stationary distribution.

The concept of invariant distribution is of key importance to the asymptotic expansion method we will use to solve stochastic volatility option pricing models. Fan [5] points out that the invariant distribution is also very important to statistical inference of stochastic volatility models. If the initial distribution is taken from the invariant density, then the process is stationary. And stationary plays an important role in time series analysis and forecasting. The structual invariablity allows people

to forecast the future based on the historical data.

Fouque [7] pointed out that the OU process is ergodic, meaning that the long-run time average of a bounded function g of the process is close to the statistical average with respect to its invariant distribution:

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t g(Y_u)du = \langle g \rangle,$$

in which $\langle g \rangle$ denotes the expectation of g under the invariant distribution.

An important concept associated with the process (3.1), or more generally, with stochastic differential equations of the form of (2.28) is *infinitesimal generator*. According to [4], an infinitesimal generator can be defined as following:

Definition 4. *Given a stochastic differential equation of the form of (2.28), the partial differential operator $\mathcal{L}$, referred to as the infinitesimal generator of X, is defined, for any function $h(x)$ with $h \in C^2(\mathbb{R})$, by*

$$\mathcal{L}h(t,x) = \alpha(t,x)\frac{\partial h}{\partial x} + \frac{1}{2}\beta^2(t,x)\frac{\partial^2 h}{\partial x^2}.$$

Fouque [7] also pointed out that the invariant distribuiton is unique for ergodic processes and can be calculated using its infinitesimal generator by solving:

$$\mathbb{E}\{\mathcal{L}g(Y_0)\} = 0$$

for any smooth and bounded g.

One final important fact about the OU process has to do with the null space of its

infinitesimal generator $\mathcal{L}$, i.e. the solutions of

$$\mathcal{L}\phi = \beta^2 \phi'' + \alpha(m - y)\phi' = 0.$$

Fouque [7] showed that the solutions to this ordinary differential equation are of the form:

$$\phi(y) = c_1 \int_{-\infty}^{y} e^{\alpha(m-z)^2/2\beta^2} dz + c_2$$

for constants c_1 and c_2. One shall be interested in solutions that are 'well-behaved', i.e. solutions that are not rapidly growing. For this reason, one may take $c_1 = 0$. So that the only admissible solutions are constant over state y. And this fact is true for ergodic Markov processes, which include OU and CIR process as special cases.

3.2   The Cox-Ingersoll-Ross Process

The Cox-Ingersoll-Ross (CIR) process is another popular process for stochastic volatility modeling which is mean-reverting. It was first proposed to model the dynamics of interest rates but also fits the purpose of volatility modeling. A CIR process is defined as the solution to the following stochastic differential equation:

$$dY_t = \kappa(m' - Y_t)dt + \eta\sqrt{Y_t}dW_t \tag{3.3}$$

in which W_t is a standard Brownian motion, κ is called the rate of mean reversion, and m' is the long-run mean level of Y. This equation does not have a closed-form solution. But the CIR process has an advantage over the OU process: the CIR process is always nonnegative. This can be intuitively seen from the defining equation that when the process approaches zero, the term multiplying dW_t vanishes and the

postive drift term $\kappa m'dt$ drives the process back into positive territory.

Although one cannot derive a closed-form solution for (3.4), the conditional distribution of Y_t can be calculated. While the computation is too long to be presented here, we only give the result: Y_t is a non-central chi-square distribution and the expectation and variance equal

$$e^{-\kappa t}y + m'(1 - e^{-\kappa t})$$

and

$$\frac{\eta^2}{\kappa}y(e^{-\kappa t} - e^{-2\kappa t}) + \frac{m'\eta^2}{2\kappa}(1 - 2e^{-\kappa t} + e^{-2\kappa t}),$$

respectively, in which y is the initial value of the process. As t goes to infinity, we have that the mean and variance of the process of its long-run distribution is m' and $\frac{m'\eta^2}{2\kappa}$, respectively.

3.3 Scales in Mean-Reverting Stochatic Volatility

Through studying the $S\&P500$ index return process, Fouque [7] found another characteristic of volatility series which is called *fast mean-reverting*. They estimated that the $S\&P500$ volatility returns to its long-run average level on a characteristic time of 1.5 day. Inspired by this phenomenon and observing that the covariance of the OU process (3.1) under its long-run distribution is

$$\mathbb{E}\{(Y_t - m)(Y_s - m)\} = \frac{\beta^2}{2\alpha}e^{-\alpha|t-s|}, \tag{3.4}$$

they assumed α to be a large constant which would cause the process Y to decorrelate quickly, a mathematical description of fast mean-reverting. They further keep $\nu^2 = \frac{\beta^2}{2\alpha}$ as fixed and write $\beta = \nu\sqrt{2\alpha}$. Since α is large, its reciprocal $\epsilon = \frac{1}{\alpha}$ is small. We will see in the next chapter that this small parameter ϵ plays the key role in the asymptotic methods for solving PDEs resulting from option pricing models in which the volatility driving process is fast mean-reverting.

For CIR process, the covariance function (3.4) becomes

$$\mathbb{E}\{(Y_t - m')(Y_s - m')\} = \frac{m'\eta^2}{2\kappa}e^{-\kappa|t-s|}. \tag{3.5}$$

Assume that $\nu^2 = \frac{m'\eta^2}{2\kappa}$ being constant, we can write $\eta = \frac{\sqrt{2\kappa}\nu}{\sqrt{m'}}$. Denote $\epsilon = \frac{1}{\kappa}$, we can further write $\eta = \frac{\sqrt{2}\nu}{\sqrt{\epsilon}\sqrt{m'}}$.

Chapter 4

The Asymptotic Expansion Method

4.1 Background and History

The asymptotic expansion method, also called the perturbation method, is used to find an approximate solution to a mathematical problem which cannot be solved exactly, by starting from the exact solution of a related problem. It leads to an expression for the desired solution in terms of a formal power series in some small parameter that quantifies the deviation from the exactly solvable problem. The leading term in the power series is the solution of the exactly solvable problem, while further terms describe the deviation in the solution, due to the deviation from the initial problem.

4.2 An Illustrative Example

We use the CIR stochastic volatility model to illustrate the asymptotic expansion method.

4.2.1 Model setup

The model we use here for illustrative purpose is similar to the model used in Fouque [7]. The difference is that we model the volatility process as a CIR process whereas Fouque used an OU process as the driving process for volatility. This enables us to get rid of the unspecified function f as that in Fouque's model, which is an non-negative function used because the OU process may take on negative values. Our model under the real-world probability measure is as the following:

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, W_t be a Brownian motion, and $\mathcal{F}_t$ the filtration associated with the Brownian motion.

$$
\begin{aligned}
dX_t &= \mu X_t dt + \sigma_t X_t dW_t, \\
\sigma_t &= \sqrt{Y_t}, \\
dY_t &= \frac{1}{\epsilon}(m - Y_t)dt + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}\sqrt{m}}\sqrt{Y_t}dZ_t^*, \\
Z_t^* &= \rho W_t + \sqrt{1 - \rho^2}Z_t,
\end{aligned}
\tag{4.1}
$$

where W_t and Z_t are independent standard Brownian motions. Note that $dW_t dZ_t^* = \rho dt$. The specification of the drift and diffusion of Y_t follows the reasoning in section 3.3.

4.2.2 Derivation of option pricing equation

We look for an equivalent probability measure under which the discounted process $e^{-rt}X_t$ is a martingale. To do this, we need the following multiple dimension version of the Girsanov's theorem:

Theorem 5 (Girsanov, multiple dimension). *Let T be a fixed positive time, and let $W_t = (W_t^{(1)}, \cdots, W_t^{(d)})$ be a d-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Assiciated with this Brownian motion, we have a filtration $\mathcal{F}_t$. Denote $\mathcal{F} = \mathcal{F}_T$. Let $\Theta = (\Theta_t^{(1)}, \cdots, \Theta_t^{(d)})$ be a d-dimensional adapted process. Define*

$$Z_t = exp\left\{ -\int_0^t \Theta_u \cdot dW_u - \frac{1}{2}\int_0^t \parallel \Theta_u \parallel^2 du \right\},$$

$$\tilde{W}_t = W_t + \int_0^t \Theta_u du,$$

and assume

$$\mathbb{E}\int_0^T \parallel \Theta_u \parallel^2 Z_u^2 du < \infty.$$

Set $Z = Z_T$. Then $\mathbb{E}Z = 1$, and under the probability measure $\tilde{\mathbb{P}}$ given by

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega)d\mathbb{P}(\omega) for\ all A \in \mathcal{F},$$

the process $\tilde{W}_t$ is a d-dimensional Brownian motion.

In the above definition,

$$\int_0^t \Theta_u \cdot dW_u = \int_0^t \sum_{j=1}^d \Theta_u^{(j)} dW_u^{(j)} = \sum_{j=1}^d \int_0^t \Theta_u^{(j)} dW_u^{(j)},$$

$\parallel \Theta_u \parallel$ denotes the Euclidean norm

$$\| \Theta_u \| = \left(\sum_{j=1}^{d} (\Theta_u^{(j)})^2 \right)^{1/2},$$

and

$$\tilde{W}_t = (\tilde{W}_t^{(1)}, \cdots, \tilde{W}_t^{(d)})$$

with

$$\tilde{W}_t^{(j)} = W_t^{(j)} + \int_0^t \Theta_u^{(j)} du, \, j = 1, \cdots, d.$$

We introduce the probability measure $\tilde{\mathbb{P}}$ using

$$\Theta_t^{(1)} = \frac{\mu - r}{\sigma_t}$$

and

$$\Theta_t^{(2)} = \gamma_t,$$

in which $\gamma_t = \gamma(Y_t)$ is an unknown function called *market price of volatility risk*. The choice of γ is not unique, thus the stochastic volatility model gives rise to an incomplete market in which the process (γ_t) parametrizes a space of equivalent measures. For each choice of γ, we denote the corresponding equivalent martingale measure by $\tilde{\mathbb{P}}^{(\gamma)}$.

Under $\tilde{\mathbb{P}}^{(\gamma)}$, model (4.1) becomes

$$dX_t = rX_t dt + \sigma_t X_t d\tilde{W}_t,$$

$$\sigma_t = \sqrt{Y_t},$$

$$dY_t = \left[\frac{1}{\epsilon}(m - Y_t) - \frac{\nu\sqrt{2}}{\sqrt{\epsilon}\sqrt{m}}\sqrt{Y_t}\left(\rho\frac{\mu - r}{Y_t} + \sqrt{1 - \rho^2}\gamma(Y_t)\right)\right]dt + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}\sqrt{m}}\sqrt{Y_t}d\tilde{Z}_t^*,$$

$$\tilde{Z}_t^* = \rho\tilde{W}_t + \sqrt{1 - \rho^2}\tilde{Z}_t.$$

$$(4.2)$$

And the corresponding risk-neutral valuation formula is

$$C(t, x, y) = \tilde{\mathbb{E}}^{(\gamma)}\{e^{-r(T-t)}(X_T - K)^+|\mathcal{F}_t\}. \tag{4.3}$$

Through an application of the Feynman-Kac theorem, we have that the function $C(t, x, y)$ should satisfy the following partial differential equation:

$$C_t + \frac{1}{2}x^2y^2C_{xx} + r(xC_x - C) + \rho\frac{\nu\sqrt{2}}{\sqrt{\epsilon}\sqrt{m}}xyC_{xy} - \frac{\nu\sqrt{2}}{\sqrt{\epsilon}\sqrt{m}}\Lambda(y)C_y + \frac{\nu^2}{\epsilon m}yC_{yy} + \frac{1}{\epsilon}(m-y)C_y = 0$$

$$(4.4)$$

in which, $\Lambda(y) = \sqrt{y}\left(\rho\frac{\mu-r}{y} + \sqrt{1 - \rho^2}\gamma(y)\right)$, and the terminal condition

$$C(T, x, y) = (x - K)^+.$$

The partial differential equation (4.4) involves terms of order $1/\epsilon$, $1/\sqrt{\epsilon}$, and 1. Introducing the following notations, we can write equation (4.4) more succinctly:

$$\mathcal{L}_0 = \frac{\nu^2}{m} y \frac{\partial^2}{\partial y^2} + (m - y) \frac{\partial}{\partial y},$$

$$\mathcal{L}_1 = \frac{\sqrt{2}\rho\nu}{\sqrt{m}} xy \frac{\partial^2}{\partial x \partial y} - \frac{\sqrt{2}\nu}{\sqrt{m}} \Lambda(y) \frac{\partial}{\partial y}, \qquad (4.5)$$

$$\mathcal{L}_2 = \frac{\partial}{\partial t} + \frac{1}{2} y^2 x^2 \frac{\partial^2}{\partial x^2} + r(x \frac{\partial}{\partial x} - \cdot),$$

and

$$\left(\frac{1}{\epsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}} \mathcal{L}_1 + \mathcal{L}_2 \right) C = 0, \qquad (4.6)$$

with terminal condition

$$C(T, x, y) = (x - K)^+. \qquad (4.7)$$

4.2.3   Asymptotic expansion of option price function

The method introduced by Fouque is to expand the solution C in powers of $\sqrt{\epsilon}$,

$$C = C_0 + \sqrt{\epsilon} C_1 + \epsilon C_2 + \epsilon \sqrt{\epsilon} C_3 + ..., \qquad (4.8)$$

where $C_0, C_1,...$are functions of (t, x, y) to be determined such that $C_0(T, x, y) = (x - K)^+$. And we only need the first two terms C_0 and C_1 to obtain a approximate option pricing formula with error bound that can be proved. The terminal condition for the second term is $C_1(T, x, y) = 0$.

Substituting (4.8) into (4.6) leads to

$$\frac{1}{\epsilon}\mathcal{L}_0 C_0 + \frac{1}{\sqrt{\epsilon}}(\mathcal{L}_0 C_1 + \mathcal{L}_1 C_0)$$

$$+ (\mathcal{L}_0 C_2 + \mathcal{L}_1 C_1 + \mathcal{L}_2 C_0)$$

$$+ \sqrt{\epsilon}(\mathcal{L}_0 C_3 + \mathcal{L}_1 C_2 + \mathcal{L}_2 C_1) \tag{4.9}$$

$$+ \cdots$$

$$= 0.$$

Equating terms of order $1/\epsilon$, we must have

$$\mathcal{L}_0 C_0 = 0. \tag{4.10}$$

Since the operator $\mathcal{L}_1$ only acts on y, we have that C_0 does not depend on y, i.e.

$$C_0 = C_0(t, x), \tag{4.11}$$

And then equating terms of $1/\sqrt{\epsilon}$, we must have

$$\mathcal{L}_0 C_1 + \mathcal{L}_1 C_0 = 0. \tag{4.12}$$

The operator $\mathcal{L}_1$ takes derivatives with respect to y but C_0 does not depend on y, so that $\mathcal{L}_1 C_0 = 0$. And thus we have $\mathcal{L}_0 C_1 = 0$. Similar to C_0 we have that C_1 also does not depend on y, so we have

$$C_1 = C_1(t, x). \tag{4.13}$$

(4.11) and (4.13) are important in that they imply that the sum of the first two terms $C_0 + \sqrt{\epsilon}C_1$ does not depend on the present volatility. The order-1 terms give

$$\mathcal{L}_0 C_2 + \mathcal{L}_1 C_1 + \mathcal{L}_2 C_0 = 0. \tag{4.14}$$

Again, since C_1 does not involve y and $\mathcal{L}_1$ takes derivatives with respect to y, we have $\mathcal{L}_1 C_1 = 0$. Thus the above equation reduces to

$$\mathcal{L}_0 C_2 + \mathcal{L}_2 C_0 = 0. \tag{4.15}$$

The variable x being fixed, $\mathcal{L}_2 C_0$ is a function of y since $\mathcal{L}_2$ involves $f(y)$. Focusing on the y dependence only, equation (4.15) is of the form

$$\mathcal{L}_0 \chi + g = 0. \tag{4.16}$$

This equation is called a *Poisson equation* for $\chi(y)$ with respect to the operator $\mathcal{L}_0$ in the variable y. It does not have solutions unless the function $g(y)$ is centered with respect to the invariant distribution of the Markov process Y. Here the centering condition implies $\langle \mathcal{L}_2 C_0 \rangle = 0$. And since C_0 does not involve y, this means $\langle \mathcal{L}_2 \rangle C_0 = 0$. Notice that $\langle \mathcal{L}_2 \rangle$ is the Black-Scholes operator with volatility parameter being the expectation of σ_t under its invariant expectation. Denoted by $\bar{\sigma}$, this expected value of σ_t is called the *effective volatility*. Therefore $C_0(t, x)$ is the solution of the Black-Scholes equation $\mathcal{L}_{BS}(\bar{\sigma}) C_0 = 0$ with terminal condition $C_0(T, x) = (x - K)^+$.

From (4.15) we also know that $\mathcal{L}_0 C_2 = -\mathcal{L}_2 C_0$. For the next step we will try to simply the expression of C_2 in order to obtain an expression for C_1. Observe that there is only one term in the operator $\mathcal{L}_2$ that involves y and C_0 does not involve y. So that $\mathcal{L}_2 C_0$ and $\langle \mathcal{L}_2 C_0 \rangle$ would differ only for one term that involves y. Also notice that we have $\langle \mathcal{L}_2 C_0 \rangle = 0$ from the centering condition, thus we can write $\mathcal{L}_0 C_2$ as

$$\mathcal{L}_0 C_2 = -(\mathcal{L}_2 C_0 - \langle \mathcal{L}_2 C_0 \rangle) = -\frac{1}{2}(y^2 - \bar{\sigma}^2)x^2 \frac{\partial^2 C_0}{\partial x^2}. \tag{4.17}$$

So that the second correcting term C_2 is given by

$$C_2(t, x, y) = -\frac{1}{2}\mathcal{L}_0^{-1}(y^2 - \bar{\sigma}^2)x^2 \frac{\partial^2 C_0}{\partial x^2}. \tag{4.18}$$

For notational convenience, we introduce a function $\phi(x)$ which solve the equation

$$\mathcal{L}_0 \phi = y^2 - \bar{\sigma}^2. \tag{4.19}$$

This function ϕ can be defined up to a difference of a contant $c(t, x)$ in terms of y. With this function, we can write C_2 as

$$C_2(t, x, y) = -\frac{1}{2}(\phi(y) + c(t, x))x^2 \frac{\partial^2 C_0}{\partial x^2}. \tag{4.20}$$

Now we can equate the term of order $\sqrt{\epsilon}$ to zero, which gives

$$\mathcal{L}_0 C_3 + \mathcal{L}_1 C_2 + \mathcal{L}_2 C_1 = 0. \tag{4.21}$$

This is again a Poisson equation for C_3 with respect to $\mathcal{L}_0$, whose solvability condition implies

$$\langle \mathcal{L}_1 C_2 + \mathcal{L}_2 C_1 \rangle = 0. \tag{4.22}$$

Using the fact that the function C_1 does not involve y and notice that $\langle \mathcal{L}_2 \rangle = \mathcal{L}_{BS}(\bar{\sigma})$, and then plug in the the expression for C_2 from (4.18),we can rewrite the above equation as

$$\langle \mathcal{L}_2 C_1 \rangle = \mathcal{L}_{BS}(\bar{\sigma}) C_1$$

$$= -\langle \mathcal{L}_1 C_2 \rangle$$

$$= \frac{1}{2} \left\langle \mathcal{L}_1 (\phi(y) + c(t,x)) x^2 \frac{\partial^2 C_0}{\partial x^2} \right\rangle$$

$$= \frac{1}{2} \left\langle \mathcal{L}_1 \phi(y) x^2 \frac{\partial^2 C_0}{\partial x^2} \right\rangle$$

$$= \frac{1}{2} \left\langle \left(\frac{\sqrt{2}\rho\nu}{\sqrt{m}} xy \frac{\partial^2}{\partial x \partial y} - \frac{\sqrt{2}\nu}{\sqrt{m}} \Lambda(y) \frac{\partial}{\partial y} \right) \left(\phi(y) x^2 \frac{\partial^2 C_0}{\partial x^2} \right) \right\rangle \qquad (4.23)$$

$$= \frac{1}{2} \left\langle \frac{\sqrt{2}\rho\nu}{\sqrt{m}} xy \frac{\partial^2}{\partial x \partial y} \left(\phi(y) x^2 \frac{\partial^2 C_0}{\partial x^2} \right) - \frac{\sqrt{2}\nu}{\sqrt{m}} \Lambda(y) \frac{\partial}{\partial y} \left(\phi(y) x^2 \frac{\partial^2 C_0}{\partial x^2} \right) \right\rangle$$

$$= \frac{1}{2} \left\langle \frac{\sqrt{2}\rho\nu}{\sqrt{m}} y\phi'(y) x \frac{\partial}{\partial x} \left(x^2 \frac{\partial^2 C_0}{\partial x^2} \right) - \frac{\sqrt{2}\nu}{\sqrt{m}} \Lambda(y) \phi' x^2 \frac{\partial^2 C_0}{\partial x^2} . \right\rangle$$

$$= \frac{\sqrt{2}}{2} \frac{\rho\nu}{\sqrt{m}} \langle y\phi'(y) \rangle x \frac{\partial}{\partial x} \left(x^2 \frac{\partial^2 C_0}{\partial x^2} \right) - \frac{\sqrt{2}}{2} \frac{\nu}{\sqrt{m}} \langle \Lambda(y) \phi' \rangle x^2 \frac{\partial^2 C_0}{\partial x^2} .$$

To further our calculation, we need to calculate the derivatives of C_0 with respect to x explicit. Recall that C_0 is the Black-Scholes formula with long-run averaged volatility $\bar{\sigma}$. It is explicitly given by

$$C_0(t,x) = xN(d_1) - Ke^{-r(T-t)} N(d_2), \qquad (4.24)$$

where K is the strike price, T is the experiation date, and

$$d_1 = \frac{ln(x/K) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = \frac{ln(x/K) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t},$$

where N denotes the distribution function of standard normal distribution.

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The first derivative of C_0 with respect to x, which is known as the *delta*, is calculated as

$$\frac{\partial C_0}{\partial x} = N(d_1). \tag{4.25}$$

The second derivative of C_0 with respect to x, which is known as the *gamma*, is calculated as

$$\frac{\partial^2 C_0}{\partial x^2} = \frac{\phi(d_1)}{x\sigma\sqrt{T-t}}, \tag{4.26}$$

in which ϕ denotes the density function of standard normal distribution.

And the third derivative of C_0 with respect to x, which is known as the *speed*, is calculated as

$$\begin{aligned}
\frac{\partial^3 C_0}{\partial x^3} &= -\frac{\phi(d_1)}{x^2\sigma\sqrt{T-t}}\left(\frac{d_1}{\sigma\sqrt{T-t}}+1\right) \\
&= -\frac{1}{x}\left(\frac{d_1}{\sigma\sqrt{T-t}}+1\right)\frac{\partial^2 C_0}{\partial x^2}.
\end{aligned} \tag{4.27}$$

Now that we have

$$x^2\frac{\partial^2 C_0}{\partial x^2} = \frac{x\phi(d_1)}{\sigma\sqrt{T-t}},$$

and thus

$$\begin{aligned}
x\frac{\partial}{\partial x}\left(x^2\frac{\partial^2 C_0}{\partial x^2}\right) &= x\left(\frac{\phi(d_1)}{\sigma\sqrt{T-t}} - \frac{\phi(d_1)d_1}{\sigma^2(T-t)}\right) \\
&= \frac{x\phi(d_1)}{\sigma\sqrt{T-t}} - \frac{x\phi(d_1)d_1}{\sigma^2(T-t)}.
\end{aligned} \tag{4.28}$$

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Observe that

$$x^3 \frac{\partial^3 C_0}{\partial x^3} = -x^2 \left(\frac{d_1}{\sigma\sqrt{T-t}} + 1 \right) \frac{\partial^2 C_0}{\partial x^2}$$

$$= -x^2 \left(\frac{d_1}{\sigma\sqrt{T-t}} + 1 \right) \frac{\phi(d_1)}{x\sigma\sqrt{T-t}} \qquad (4.29)$$

$$= -\frac{x\phi(d_1)}{\sigma\sqrt{T-t}} - \frac{x\phi(d_1)d_1}{\sigma^2(T-t)}.$$

So that

$$x \frac{\partial}{\partial x} \left(x^2 \frac{\partial^2 C_0}{\partial x^2} \right) = x^3 \frac{\partial^3 C_0}{\partial x^3} + 2 \frac{x\phi(d_1)}{\sigma\sqrt{T-t}}$$

$$= x^3 \frac{\partial^3 C_0}{\partial x^3} + 2x^2 \frac{\partial^2 C_0}{\partial x^2}. \qquad (4.30)$$

Substituting the second equation of (4.30) into (4.23), we have

$$\langle \mathcal{L}_2 C_1 \rangle = \mathcal{L}_{BS}(\bar{\sigma}) C_1$$

$$= \frac{\sqrt{2}}{2} \frac{\rho\nu}{\sqrt{m}} \langle y\phi'(y) \rangle x \frac{\partial}{\partial x} \left(x^2 \frac{\partial^2 C_0}{\partial x^2} \right) - \frac{\sqrt{2}}{2} \frac{\nu}{\sqrt{m}} \langle \Lambda(y)\phi'(y) \rangle x^2 \frac{\partial^2 C_0}{\partial x^2}$$

$$= \frac{\sqrt{2}}{2} \frac{\rho\nu}{\sqrt{m}} \langle y\phi'(y) \rangle \left(x^3 \frac{\partial^3 C_0}{\partial x^3} + 2x^2 \frac{\partial^2 C_0}{\partial x^2} \right) - \frac{\sqrt{2}}{2} \frac{\nu}{\sqrt{m}} \langle \Lambda(y)\phi'(y) \rangle x^2 \frac{\partial^2 C_0}{\partial x^2}$$

$$= \frac{\sqrt{2}}{2} \frac{\rho\nu}{\sqrt{m}} \langle y\phi'(y) \rangle x^3 \frac{\partial^3 C_0}{\partial x^3} + \left(\sqrt{2} \frac{\rho\nu}{\sqrt{m}} \langle y\phi'(y) \rangle - \frac{\sqrt{2}}{2} \frac{\nu}{\sqrt{m}} \langle \Lambda(y)\phi'y \rangle \right) x^2 \frac{\partial^2 C_0}{\partial x^2},$$

$$(4.31)$$

with terminal condition $C_1(T,x) = 0$.

Now it is convenient to donote the first correction term $\sqrt{\epsilon} C_1(t,x)$ by $\tilde{C}_1(t,x)$ and rewrite the RHS of the last equation in (4.31) in terms of $H(t,x)$, V_2, V_3, which are

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given by

$$H(t,x) = \left(V_2 x^2 \frac{\partial^2 C_0}{\partial x^2} + V_3 x^3 \frac{\partial^3 C_0}{\partial x^3} \right),$$

$$V_2 = \frac{\nu}{\sqrt{2\alpha}\sqrt{m}} (2\rho \langle y\phi'(y) \rangle - \langle \Lambda(y)\phi'(y) \rangle), \qquad (4.32)$$

$$V_3 = \frac{\rho\nu}{\sqrt{2\alpha}\sqrt{m}} \langle y\phi'(y) \rangle.$$

Multiplying both sides of (4.31) by $\sqrt{\epsilon} = \frac{1}{\sqrt{\alpha}}$, we have

$$\mathcal{L}_{BS}(\bar\sigma)\tilde{C}_1 = -(T-t)H(t,x) = V_2 x^2 \frac{\partial^2 C_0}{\partial x^2} + V_3 x^3 \frac{\partial^3 C_0}{\partial x^3}. \qquad (4.33)$$

Using the fact that the operator $x^m \frac{\partial^m}{\partial x^m}$ commutes with the operator $x^n \frac{\partial^n}{\partial x^n}$, where m and n are positive intergers, we have that the operator $x^m \frac{\partial^m}{\partial x^m}$ commutes with $\mathcal{L}_{BS}(\bar\sigma)$. This observation is important in that it enables us to write $\tilde{C}_1$ explicitly as

$$\tilde{C}_1(t,x) = -(T-t)\left(V_2 x^2 \frac{\partial^2 C_0}{\partial x^2} + V_3 x^3 \frac{\partial^3 C_0}{\partial x^3} \right). \qquad (4.34)$$

To check this, we see that

$$\begin{aligned}
&\mathcal{L}_{BS}(\bar\sigma)(-(T-t)H(t,x)) \\
&= \left(\frac{\partial}{\partial t} + \frac{1}{2}x^2 y^2 \frac{\partial^2}{\partial x^2} + r(x\frac{\partial}{\partial x} - \cdot) \right)(-(T-t)H(t,x)) \\
&= H(t,x) - (T-t)\mathcal{L}_{BS}(\bar\sigma)\left(V_2 x^2 \frac{\partial^2 C_0}{\partial x^2} + V_3 x^3 \frac{\partial^3 C_0}{\partial x^3} \right) \qquad (4.35)\\
&= H(t,x) - (T-t)\left(V_2 x^2 \frac{\partial^2}{\partial x^2}(\mathcal{L}_{BS}(\bar\sigma)C_0) + V_3 x^3 \frac{\partial^3}{\partial x^3}(\mathcal{L}_{BS}(\bar\sigma)C_0) \right) \\
&= H(t,x),
\end{aligned}$$

in which we use the fact that $\mathcal{L}_{BS}(\bar\sigma)C_0 = 0$.

Now we can give first-order corrected pricing formula. Denoting the exact solution to the equation (4.6) with terminal condition (4.7) by $\tilde{C}_{BS}$, since it is not the Black-Scholes price function but closely related to the Black-Scholes function, we have:

$$\tilde{C}_{BS} \approx C_0 + \tilde{C}_1 = C_0 - (T - t)\left(V_2 x^2 \frac{\partial^2 C_0}{\partial x^2} + V_3 x^3 \frac{\partial^3 C_0}{\partial x^3}\right), \qquad (4.36)$$

in which C_0 is the Black-Scholes price with volatility parameter equals the long-run averaged volatility, which is essentially the expected value of the process Y_t given in (4.1) with respect to its invariant distribution.

If substituting in the expressions of $x^2 \frac{\partial^2 C_0}{\partial x^2}$ and $x^3 \frac{\partial^3 C_0}{\partial x^3}$, we can further write $\tilde{C}_1$ as

$$\tilde{C}_1 = \frac{x\phi(d_1)}{\bar{\sigma}}\left[(V_3 - V_2)\sqrt{T - t} + V_3 \frac{d_1}{\bar{\sigma}}\right]. \qquad (4.37)$$

Expression (4.37) will be useful when we show how the parameters V_2 and V_3 can be calibrated from implied volatility data in the next section.

4.2.4   Implied volatility and calibration

The implied volatility, denoted by I, is initially defined as the value of volatility parameter that will equate the Black-Scholes pricing formula to the option price observed from the market. Mathematically, we write

$$C_{BS}(t, x; K, T; I) = C^{observed}(t, x; K, T). \qquad (4.38)$$

When dealing with generalized option pricing models, like the one we have here, the definition of implied volatility should be modified as the value of volatility parameter

which will equate the theoretical pricing formula to the option price observed from the market. In our setting, we have

$$\tilde{C}_{BS}(t, x; K, T; I) = C^{observed}(t, x; K, T). \tag{4.39}$$

To exploit the information contained in the implied volatility data, we first expand the implied volatility I in powers of $\sqrt{\epsilon}$ around the long-run avaraged volatility $\bar{\sigma}$:

$$I = \bar{\sigma} + \sqrt{\epsilon} I_1 + \epsilon I_2 + \cdots . \tag{4.40}$$

Then we take the Taylor expansion of the model theoretical option price function $\tilde{C}_{BS}$ in its volatility parameter around the long-run averaged volatility $\bar{\sigma}$:

$$
\begin{aligned}
\tilde{C}_{BS}(t, x; K, T; I) &= \tilde{C}_{BS}(t, x; K, T; \bar{\sigma}) + (I - \bar{\sigma}) \frac{\partial \tilde{C}_{BS}}{\partial \sigma}(t, x; K, T; \bar{\sigma}) \\
&\quad + (I - \bar{\sigma})^2 \frac{\partial^2 \tilde{C}_{BS}}{\partial \sigma^2}(t, x; K, T; \bar{\sigma}) + \cdots .
\end{aligned}
\tag{4.41}
$$

Substituting (4.40) into (4.41), we have

$$
\begin{aligned}
\tilde{C}_{BS}(t, x; K, T; I) &= \tilde{C}_{BS}(t, x; K, T; \bar{\sigma}) + \sqrt{\epsilon} I_1 \frac{\partial \tilde{C}_{BS}}{\partial \sigma}(t, x; K, T; \bar{\sigma}) \\
&\quad + \epsilon \left(I_2 \frac{\partial \tilde{C}_{BS}}{\partial \sigma}(t, x; K, T; \bar{\sigma}) + I_1^2 \frac{\partial^2 \tilde{C}_{BS}}{\partial \sigma^2}(t, x; K, T; \bar{\sigma}) \right) + \cdots \\
&= \tilde{C}_{BS}(t, x; K, T; \bar{\sigma}) + \sqrt{\epsilon} I_1 \frac{\partial \tilde{C}_{BS}}{\partial \sigma}(t, x; K, T; \bar{\sigma}) + \mathcal{O}(\epsilon).
\end{aligned}
\tag{4.42}
$$

On the other hand, we have

$$\tilde{C}_{BS} = C_0 + \sqrt{\epsilon} C_1 + \epsilon C_2 + ..., \tag{4.43}$$

and $C_0 = C_{BS}(t, x; K, T; \bar{\sigma})$. So that we can equate the order ϵ terms in (4.42) and (4.43) and have

$$\sqrt{\epsilon}I_1 \frac{\partial \tilde{C}_{BS}}{\partial \sigma}(t, x; K, T; \bar{\sigma}) = \sqrt{\epsilon}C_1. \tag{4.44}$$

or

$$\sqrt{\epsilon}I_1 = \sqrt{\epsilon}C_1 \left[\frac{\partial \tilde{C}_{BS}}{\partial \sigma}(t, x; K, T; \bar{\sigma}) \right]^{-1}. \tag{4.45}$$

We will see that $\frac{\partial \tilde{C}_{BS}}{\partial \sigma}(t, x; K, T; \bar{\sigma})$ contains information about V_2 and V_3, which are the parameters needed for option pricing. So that calibrating the implied volatility expansion to market data up to the accuracy of order $\sqrt{\epsilon}$ will be enough for option pricing purpose.

Substituting (4.45) into (4.40) gives

$$I = \bar{\sigma} + \sqrt{\epsilon}C_1 \left[\frac{\partial \tilde{C}_{BS}}{\partial \sigma}(t, x; K, T; \bar{\sigma}) \right]^{-1} + \mathcal{O}(\epsilon). \tag{4.46}$$

The partial derivative of the Black-Scholes pricing function with respect to its volatility parameter, known as *vegga*, can be calculated explicitly as

$$\begin{aligned}
\frac{\partial C_{BS}}{\partial \sigma} &= x\phi(d_1)\sqrt{T - t} \\
&= \frac{xe^{-d_1^2/2}\sqrt{T - t}}{\sqrt{2\pi}}.
\end{aligned} \tag{4.47}$$

Substituting (4.47) together with the expression (4.37) for $\tilde{C}_1 = \sqrt{\epsilon}C_1$ into (4.46), we have

$$I = \bar{\sigma} + \frac{x\phi(d_1)}{\bar{\sigma}}\left[(V_3 - V_2)\sqrt{T-t} + V_3\frac{d_1}{\bar{\sigma}}\right]\frac{1}{x\phi(d_1)\sqrt{T-t}} + \mathcal{O}(\epsilon)$$

$$= \bar{\sigma} + \frac{1}{\bar{\sigma}\sqrt{T-t}}\left[(V_3 - V_2)\sqrt{T-t} + V_3\frac{d_1}{\bar{\sigma}}\right] + \mathcal{O}(\epsilon)$$

$$= \bar{\sigma} + \frac{V_3 d_1}{\bar{\sigma}^2\sqrt{T-t}} + \frac{V_3 - V_2}{\bar{\sigma}} + \mathcal{O}(\epsilon) \tag{4.48}$$

$$= \bar{\sigma} + \frac{V_3}{\bar{\sigma}^3}\left[\frac{\ln(x/K)}{T-t}\right] + \frac{V_3}{\bar{\sigma}^3}(r + \frac{\bar{\sigma}^2}{2}) + \frac{V_3 - V_2}{\bar{\sigma}} + \mathcal{O}(\epsilon)$$

$$= -\frac{V_3}{\bar{\sigma}^3}\left[\frac{\ln(K/x)}{T-t}\right] + \bar{\sigma} + \frac{V_3}{\bar{\sigma}^3}(r + \frac{3}{2}\bar{\sigma}^2) - \frac{V_2}{\bar{\sigma}} + \mathcal{O}(\epsilon).$$

This shows that the implied volatility function is an affine function of the log-moneyness-to-maturity ratio (LMMR) up to order $\mathcal{O}(\epsilon)$.

Denote

$$a = -\frac{V_3}{\bar{\sigma}^3},$$
$$b = \bar{\sigma} + \frac{V_3}{\bar{\sigma}^3}(r + \frac{3}{2}\bar{\sigma}^2) - \frac{V_2}{\bar{\sigma}}, \tag{4.49}$$

we can express V_2 and V_3 as

$$V_2 = \bar{\sigma}((\bar{\sigma} - b) - a(r + \frac{3}{2}\bar{\sigma}^2)),$$
$$V_3 = -a\bar{\sigma}^3. \tag{4.50}$$

Since a and b can be calibrated from implied volatility data, the corrected option pricing formula (4.36) can be evaluated.

4.3 Remarks

In [7, Fouque,2000], they use the model

$$dX_t = \mu X_t dt + \sigma_t X_t dW_t,$$

$$\sigma_t = f(Y_t),$$

$$dY_t = \alpha(m - Y_t) + \beta d\hat{Z}_t, \tag{4.51}$$

$$\hat{Z}_t = \rho W_t + \sqrt{1 - \rho^2} Z_t,$$

in which W_t and Z_t are independent Brownian motions and f is an unknown non-negative function. The corresponding option price formula is

$$\tilde{C}(t, x, y) = C_0 - (T - t) \left(V_2 x^2 \frac{\partial^2 C_0}{\partial x^2} - V_3 x^3 \frac{\partial^3 C_0}{\partial x^3} \right). \tag{4.52}$$

And the implied volatility expansion is the same as (4.48) with V_2 and V_3 given by

$$V_2 = \frac{\nu}{2\alpha} (2\rho \langle f \phi' \rangle - \langle \Lambda \phi' \rangle),$$

$$V_3 = \frac{\rho \nu}{\sqrt{2\alpha}} \langle f \phi' \rangle, \tag{4.53}$$

where $\Lambda(y) = \rho \frac{\mu - r}{f(y)} + \gamma \sqrt{1 - \rho^2}$ and ϕ solves the equation

$$\mathcal{L}_0 \phi = f^2(y) - \langle f^2 \rangle.$$

For the model (4.51), Fouque also showed for smooth and bounded terminal conditions (which the European call option does not satisfy) that the difference between the first order approximation and the exact solution of the problem is controlled by a constant times ϵ, in which the constant is independent of ϵ but may depend on y,

the current state of the volatility driving process.

Chapter 5

Multiscale Expansion

In [9, Fouque,2003], a multisacle version of the asymptotic expansion method is developed. The multiscale expansion method is able to deal with models in which volatility is driven by two stochastic processes, both of whcih are fast mean-reverting. And the two processes are allowed to run on different scales.

In this chapter, we propose a relatively simple model in which volatility is modeled as a linear combination of two CIR processes. This model is motivated by the *factor model* in portfolio optimization. We assume that asset volatility consists two parts: market volatility and asset-specific volatility. In reality, the VIX index can serve as the market volatility component and the remaining part can serve as the asset-specific volatility component. Using the multiscale expansion method, we come up with an implied volatility function with richer structure than that in the single-scale case.

5.1 A Volatility 'Factor Model'

We consider the following two-factor stochastic volatility model uder the real world measure:

$$dX_t = \mu X_t dt + \sigma_t X_t dW_t^{(0)},$$

$$\sigma_t = \beta_1 Y_t + \beta_2 Z_t,$$

$$dY_t = \frac{1}{\epsilon}(m_1 - Y_t)dt + \frac{\nu_1\sqrt{2}}{\sqrt{\epsilon}\sqrt{m_1}}\sqrt{Y_t}dW_t^{(1)}, \tag{5.1}$$

$$dZ_t = \frac{1}{\delta}(m_2 - Z_t)dt + \frac{\nu_2\sqrt{2}}{\sqrt{\delta}\sqrt{m_2}}\sqrt{Z_t}dW_t^{(2)},$$

where $W_t^{(0)}$, $W_t^{(1)}$ and $W_t^{(2)}$ are standard Brownian motions whose correlation structures are given by

$$d\langle W_t^{(0)}, W_t^{(1)}\rangle = \rho_1 dt,$$

$$d\langle W_t^{(0)}, W_t^{(2)}\rangle = \rho_2 dt, \tag{5.2}$$

$$d\langle W_t^{(1)}, W_t^{(2)}\rangle = \rho_{12} dt.$$

Through an application of Girsanov's theorem, and assuming that the market price of risk functions are zero, we obtain the model under the risk-neutral probability measure:

$$dX_t = rX_t dt + \sigma_t X_t d\tilde{W}_t^{(0)},$$

$$\sigma_t = \beta_1 Y_t + \beta_2 Z_t,$$

$$dY_t = \frac{1}{\epsilon}(m_1 - Y_t)dt + \frac{\nu_1\sqrt{2}}{\sqrt{\epsilon}\sqrt{m_1}}\sqrt{Y_t}d\tilde{W}_t^{(1)},$$

$$dZ_t = \frac{1}{\delta}(m_2 - Z_t)dt + \frac{\nu_2\sqrt{2}}{\sqrt{\delta}\sqrt{m_2}}\sqrt{Z_t}d\tilde{W}_t^{(2)},$$

$$(5.3)$$

where $\tilde{W}_t^{(0)}$, $\tilde{W}_t^{(1)}$ and $\tilde{W}_t^{(2)}$ having the same correlation structure as $W_t^{(0)}$, $W_t^{(1)}$ and $W_t^{(2)}$.

5.2 The Pricing PDE

By the No Arbitrage argument, we have the call option price under our model should satisfy the following partial differential equation:

$$\mathcal{L}^{\epsilon,\delta}C^{\epsilon,\delta}(t,x,y,z,q,u) = 0, t < T,$$

$$C^{\epsilon,\delta}(T,x,y,z,q,u) = (x - K)^+,$$

$$(5.4)$$

where the operator $\mathcal{L}^{\epsilon,\delta}$ is given by

$$\mathcal{L}^{\epsilon,\delta} \triangleq \frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2 + \frac{1}{\sqrt{\delta}}\mathcal{M}_1 + \frac{1}{\delta}\mathcal{M}_2 + \frac{1}{\sqrt{\epsilon\delta}}\mathcal{M}_3,$$

in which

$$\mathcal{L}_0 = (m_1 - y)\frac{\partial}{\partial y} + \frac{\nu_1^2}{m_1}y\frac{\partial^2}{\partial y^2},$$

$$\mathcal{L}_1 = \frac{\nu_1\sqrt{2}}{\sqrt{m_1}}\rho_1(\beta_1 y + \beta_2 z)yx\frac{\partial^2}{\partial x\partial y},$$

$$\mathcal{L}_2 = \frac{\partial}{\partial t} + \frac{1}{2}(\beta_1 y + \beta_2 z)^2 x^2 \frac{\partial^2}{\partial x^2} + r\left(x\frac{\partial}{\partial x} - \cdot\right),$$

$$\mathcal{M}_1 = \frac{\nu_2\sqrt{2}}{\sqrt{m_2}}\rho_2(\beta_1 y + \beta_2 z)zx\frac{\partial^2}{\partial x\partial z},$$

$$\mathcal{M}_2 = (m_2 - z)\frac{\partial}{\partial z} + \frac{\nu_2^2}{m_2}z\frac{\partial^2}{\partial z^2},$$

$$\mathcal{M}_3 = \frac{4\nu_1\nu_2\rho_{12}}{\sqrt{m_1 m_2}}\sqrt{yz}\frac{\partial^2}{\partial y\partial z}.$$

(5.5)

Here, the operator $\mathcal{L}_2$ is the Black-Scholes operator with volatility parameter equal $\beta_1 y + \beta_2 z$. $\mathcal{L}_0$ and $\mathcal{M}_2$ are the infinitesimal generators of the CIR processes Y_t and Z_t, respectively. $\mathcal{L}_1$ and $\mathcal{M}_3$ contain the mixed partial derivative due to the correlation between Brownian motions driving the stock price and volatility factors.

5.3   Asymptotic Expansion

We first consider an expansion of the price in the powers of $\sqrt{\delta}$,

$$C^{\epsilon,\delta} = C_0^\epsilon + \sqrt{\delta}C_1^\epsilon + \delta C_2^\epsilon + \cdots ,$$

(5.6)

According to [9], the leading order term C_0^ϵ is defined as the unique solution to the following boundary value problem

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$$\left(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2\right)C_0^\epsilon = 0 \tag{5.7}$$

$$C_0^\epsilon(T, x, y) = h(x).$$

and the term C_1^ϵ is defined as the unique solution to the boundary value problem

$$\left(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2\right)C_1^\epsilon = -\left(\mathcal{M}_1 + \frac{1}{\sqrt{\epsilon}}\mathcal{M}_3\right)C_0^\epsilon \tag{5.8}$$

$$C_1^\epsilon(T, x, y) = 0.$$

Techniques available so far are only able to calculate the first correction C_1^ϵ.

Next we will expand C_0^ϵ and C_1^ϵ in powers of $\sqrt{\epsilon}$, the square root of the fast scale, to obtain an approximation for the price $C^{\epsilon,\delta}$. Specifically, we will consider expansions

$$C_k^\epsilon = C_{0,k} + \sqrt{\epsilon}C_{1,k} + \epsilon C_{2,k} + \cdots \text{ for all } k \in \mathbb{N}.$$

5.3.1 Expansion in ϵ

The expansion of the first term C_0^ϵ gives

$$C_0^\epsilon = C_0 + \sqrt{\epsilon}C_{1,0} + \epsilon C_{2,0} + \epsilon^{3/2}C_{3,0} + \cdots \tag{5.9}$$

Plug (5.9) into (5.8) we have

$$\left(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2\right)\left(C_0 + \sqrt{\epsilon}C_{1,0} + \epsilon C_{2,0} + \epsilon^{3/2}C_{3,0} + \cdots\right) = 0$$

which gives

$$\frac{1}{\epsilon}\mathcal{L}_0 C_0 + \frac{1}{\sqrt{\epsilon}}\left(\mathcal{L}_0 C_{1,0} + \mathcal{L}_1 C_0\right) + \left(\mathcal{L}_0 C_{2,0} + \mathcal{L}C_{1,0} + \mathcal{L}_2 C_0\right) + \cdots = 0$$

Find the equations associated with the first two leading terms are

$$\mathcal{L}_0 C_0 = 0 \tag{5.10}$$

$$\mathcal{L}_0 C_{1,0} + \mathcal{L}_0 C_0 = 0 \tag{5.11}$$

These are two homogeneous partical differential equation in y and q and we therefore take

$$C_0 = C_0(t, x, z, u)$$

and

$$C_{1,0} = C_{1,0}(t, x, z, u).$$

Note that the order one terms give

$$\mathcal{L}_0 C_{2,0} + \mathcal{L}_2 C_0 = 0 \tag{5.12}$$

since $\mathcal{L}_0 C_{1,0} = 0$. This is a Poisson equation in $C_{2,0}$ w.r.t the variables y and q. And there will be no solutions unless $\mathcal{L}_2 C_0$ is in the orthogonal complement of the null space of $\mathcal{L}_0^*$, which is called the Fredholm alternative of $\mathcal{L}_0$. This is equivalent to saying that $\mathcal{L}_2 C_0$ has mean zero w.r.t the invariant measure of the CIR processe Y_t, i.e. $\langle \mathcal{L}_2 C_0 \rangle = 0$. Here the bracket notation means integartion w.r.t the invariant distribution of the CIR process Y. Because C_0 does not depend on y or q, we have

$$\langle \mathcal{L}_2 C_0 \rangle = \langle \mathcal{L}_2 \rangle C_0$$

where

$$\langle \mathcal{L}_2 \rangle = \frac{\partial}{\partial t} + \left(\frac{1}{2} \beta_1^2 \langle y^2 \rangle + \beta_1 \beta_2 \langle y \rangle z + \frac{1}{2} \beta_2^2 z^2 \right) x^2 \frac{\partial^2}{\partial x^2} + r \left(x \frac{\partial}{\partial x} - \cdot \right),$$

the Black-Scholes operator with volatility

$$\langle \sigma^2(y, z) \rangle := \bar{\sigma}^2(z),$$

where $\bar{\sigma}(z)$ is introduced for notational convenience.

According to [9], the function C_0 is defined as the solution to the following boundary value problem

$$\langle \mathcal{L}_2 \rangle C_0 = 0$$
$$C_0(T, x, z, u) = h(x) \tag{5.13}$$

Thus we have

$$C_0(t, x, z, u) = C_{BS}\left(t, x; \beta \bar{\sigma}^2(z) + \bar{f}(u), \bar{\sigma}(z) \right)$$

with C_{BS} being the Black-Scholes pricing function.

Next we derive an expression for $C_{1,0}$. From the Poisson equation (5.12) and the associated centering condition we deduce that

$$C_{2,0} = -\mathcal{L}_0^{-1}(\mathcal{L}_2 C_0)$$
$$= -\mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle)C_0 \tag{5.14}$$

up to an additive function which does not depend on y and which will not play a role in the problem that defines $C_{1,0}$. The next order term in the ϵ expansion gives the following Poisson equation in $C_{3,0}$:

$$\mathcal{L}_0 C_{3,0} + \mathcal{L}_1 C_{2,0} + \mathcal{L}_2 C_{1,0} = 0 \tag{5.15}$$

The centering condition for this equation

$$\langle \mathcal{L}_1 C_{2,0} + \mathcal{L}_2 C_{1,0} \rangle = 0$$

gives the following problem that defines $C_{1,0}$:

$$\langle \mathcal{L}_2 \rangle C_{1,0} = -\langle \mathcal{L}_1 C_{2,0} \rangle$$
$$= -\langle \mathcal{L}_1(-\mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle)C_0) \rangle$$
$$= \langle \mathcal{L}_1 \mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) \rangle C_0 \tag{5.16}$$
$$\triangleq \mathcal{A}C_0$$
$$C_{1,0}(T, x, z, u) = 0$$

The function $C_{1,0}$ can in fact be written as

$$C_{1,0} = -(T - t)\mathcal{A}C_0. \tag{5.17}$$

To see this, we compute the opreators explicitly. To facilitate calculation, we intro-

duce a function $\phi(y, z)$ which solves the following Poisson equation:

$$\mathcal{L}_0\phi(y, z) = \mathcal{L}_2 - \langle\mathcal{L}_2\rangle$$

$$= \left[\frac{1}{2}\beta_1^2(y^2 - \langle y^2\rangle) + \beta_1\beta_2(y - \langle y\rangle)z\right]x^2\frac{\partial^2}{\partial x^2} \tag{5.18}$$

$$= \left[\frac{1}{2}\beta_1^2(y^2 - \nu_1^2 - m_1) + \beta_1\beta_2(y - m_1)z\right]x^2\frac{\partial^2}{\partial x^2}.$$

Note that ϕ is defined up to an additive function that depends only on the variables z, which will not affect $\mathcal{A}$. With these notions, we have

$$\mathcal{A} = \langle\mathcal{L}_1\mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle\mathcal{L}_2\rangle)\rangle$$

$$= \langle\mathcal{L}_1\phi(y, z)\rangle \tag{5.19}$$

$$= \frac{\nu_1\sqrt{2}\rho_1}{\sqrt{m_1}}\left(\beta_1\langle y^2\frac{\partial\phi}{\partial y}\rangle + \beta_2 z\langle y\frac{\partial\phi}{\partial y}\rangle\right)x\frac{\partial}{\partial x}\left(x^2\frac{\partial^2}{\partial x^2}\right).$$

Using the fact that the operator $x^m\frac{\partial^m}{\partial x^m}$ commutes with the operator $x^n\frac{\partial^n}{\partial x^n}$, which implies that the operator $x^m\frac{\partial^m}{\partial x^m}$ commutes with $\langle\mathcal{L}_2\rangle$, we can verify that the function $C_{1,0}$ ginven by (5.17) does solve problem (5.16):

$$\langle\mathcal{L}_2\rangle C_{1,0} = \langle\mathcal{L}_2\rangle\left(-(T-t)\mathcal{A}C_0\right)$$

$$= -\left(\langle\mathcal{L}_2\rangle(T-t)\right)\mathcal{A}C_0 - (T-t)\langle\mathcal{L}_2\rangle\mathcal{A}C_0 \tag{5.20}$$

$$= \mathcal{A}C_0.$$

5.3.2   Expansion of C_1^ϵ

We next carry out the expansion of C_1^ϵ in powers of $\sqrt{\epsilon}$. We write

$$C_1^\epsilon = C_{0,1} + \sqrt{\epsilon}C_{1,1} + \epsilon C_{2,1} + \epsilon^{3/2}C_{3,1} + \cdots \tag{5.21}$$

and derive below an explicit expression for $C_{0,1}$. Substituting the expansion of C_1^ϵ into (5.8), and equating the order $1/\epsilon$ terms, we have

$$\mathcal{L}_0 C_{0,1} = 0$$

As before, this implies that the function $C_{0,1}$ does not depend on the variable y. The next order terms give

$$\mathcal{L}_0 C_{1,1} + \mathcal{L}_1 C_{0,1} = -\mathcal{M}_3 C_0$$

Note that $\mathcal{L}_1$ takes derivatives w.r.t y and q whereas $C_{0,1}$ does not involve y. So that $\mathcal{L}_1 C_{0,1} = 0$. For the same reason we have that $\mathcal{M}_3 C_0 = 0$. Consequently we have

$$\mathcal{L}_0 C_{1,1} = 0,$$

which implies that $C_{1,1} = C_{1,1}(t, x, z)$, the same as $C_{1,0}$ and $C_{0,1}$.

Evaluating the terms of order one, we have

$$\mathcal{L}_0 C_{2,1} + \mathcal{L}_1 C_{1,1} + \mathcal{L}_2 C_{0,1} = -\mathcal{M}_1 C_0 - \mathcal{M}_3 C_{1,0}.$$

Using the facts that $\mathcal{L}_1 C_{1,1} = 0$ and $\mathcal{M}_3 C_{1,0} = 0$, we have

$$\mathcal{L}_0 C_{2,1} + \mathcal{L}_2 C_{0,1} = -\mathcal{M}_1 C_0.$$

This is a Poisson equation in y for $C_{2,1}$, and the associated solvability condition gives the following defining equation for $C_{0,1}$:

$$\langle \mathcal{L}_2 \rangle C_{0,1} = -\langle \mathcal{M}_1 \rangle C_0$$

$$C_{0,1}(T, x, z, u) = 0$$

(5.22)

Observe the fact that

$$
\begin{aligned}
\langle \mathcal{M}_1 \rangle &= \left\langle \frac{\nu_2 \sqrt{2}}{\sqrt{m_2}} \rho_2 (\beta_1 y + \beta_2 z) zx \frac{\partial^2}{\partial x \partial z} \right\rangle \\
&= \left[\frac{\nu_2 \sqrt{2}}{\sqrt{m_2}} \rho_2 (\beta_1 m_1 + \beta_2 z) zx \frac{\partial}{\partial x} \right] \frac{\partial}{\partial z} \\
&\triangleq M_1 \frac{\partial}{\partial z},
\end{aligned}
$$

(5.23)

and that

$$
\begin{aligned}
\frac{\partial C_0}{\partial z} &= \frac{\partial C_0}{\partial \sigma} \frac{\partial \sigma}{\partial z} \\
&= (T - t) \bar{\sigma}(z) \bar{\sigma}'(z) x^2 \frac{\partial C_0}{\partial x^2},
\end{aligned}
$$

(5.24)

we have that $C_{0,1}$ can be written as

$$C_{0,1} = \frac{(T - t)}{2} \langle \mathcal{M}_1 \rangle C_0.$$

(5.25)

To check this, we have

$$\langle \mathcal{L}_2 \rangle C_{0,1} = \langle \mathcal{L}_2 \rangle \left[\frac{(T-t)}{2} \langle \mathcal{M}_1 C_0 \rangle \right]$$

$$= \langle \mathcal{L}_2 \rangle \left[\frac{(T-t)^2}{2} M_1 \bar{\sigma}(z) \bar{\sigma}'(z) x^2 \frac{\partial C_0}{\partial x^2} \right]$$

$$= \left(\langle \mathcal{L}_2 \rangle \frac{(T-t)^2}{2} \right) M_1 \bar{\sigma}(z) \bar{\sigma}'(z) x^2 \frac{\partial C_0}{\partial x^2} + \frac{(T-t)^2}{2} \left(\langle \mathcal{L}_2 \rangle M_1 \bar{\sigma}(z) \bar{\sigma}'(z) x^2 \frac{\partial C_0}{\partial x^2} \right)$$

$$= -(T-t) M_1 \bar{\sigma}(z) \bar{\sigma}'(z) x^2 \frac{\partial C_0}{\partial x^2} + \frac{(T-t)^2}{2} \left(M_1 \bar{\sigma}(z) \bar{\sigma}'(z) x^2 \frac{\partial}{\partial x^2} \langle \mathcal{L}_2 \rangle C_0 \right)$$

$$= -\mathcal{M}_1 C_0.$$

$$(5.26)$$

5.3.3 Price approximation

From the expansion of $C^{\epsilon,\delta}, C_0^{\epsilon}$ and C_1^{ϵ} in (5.6), (5.9) and (5.21), respectively, we deduce that the call option price in our model can be approximated as

$$C^{\epsilon,\delta} \approx \tilde{C}^{\epsilon,\delta} \triangleq C_0 + \sqrt{\epsilon} P_{1,0} + \sqrt{\delta} P_{0,1}$$

$$= C_0 - (T-t) \left(\sqrt{\epsilon} \mathcal{A} - \frac{\sqrt{\delta}}{2} \langle \mathcal{M}_1 \rangle \right) C_0,$$

$$= C_0 - (T-t) \left(\sqrt{\epsilon} \frac{\nu_1 \sqrt{2} \rho_1}{\sqrt{m_1}} \left(\beta_1 \langle y^2 \frac{\partial \phi}{\partial y} \rangle + \beta_2 z \langle y \frac{\partial \phi}{\partial y} \rangle \right) x \frac{\partial}{\partial x} \left(x^2 \frac{\partial^2}{\partial x^2} \right) \right.$$

$$\left. - \frac{\sqrt{\delta}}{2} \frac{\nu_2 \sqrt{2} \rho_2}{\sqrt{m_2}} (\beta_1 m_1 + \beta_2 z) z x \frac{\partial^2}{\partial x \partial z} \right) C_0$$

$$= C_0 - \left(\sqrt{\epsilon} \frac{\nu_1 \sqrt{2} \rho_1}{\bar{\sigma} \sqrt{m_1}} \left(\beta_1 \langle y^2 \frac{\partial \phi}{\partial y} \rangle + \beta_2 z \langle y \frac{\partial \phi}{\partial y} \rangle \right) x \frac{\partial}{\partial x} \right.$$

$$\left. - (T-t) \frac{\sqrt{\delta}}{2} \frac{\nu_2 \sqrt{2} \rho_2}{\sqrt{m_2}} (\beta_1 m_1 + \beta_2 z) z \bar{\sigma}'(z) x \frac{\partial}{\partial x} \right) \frac{\partial C_0}{\partial \sigma}$$

$$(5.27)$$

in which we used the facts that

$$\frac{\partial C_0}{\partial z} = \frac{\partial C_0}{\partial \sigma} \bar{\sigma}'(z) \tag{5.28}$$

and

$$x^2 \frac{\partial^2 C_0}{\partial x^2} = \frac{1}{\bar{\sigma}(T-t)} \frac{\partial C_0}{\partial \sigma}. \tag{5.29}$$

Introducing notations

$$
\begin{aligned}
V^\epsilon &= \sqrt{\epsilon} \frac{\nu_1 \sqrt{2} \rho_1}{\bar{\sigma} \sqrt{m_1}} \left(\beta_1 \langle y^2 \frac{\partial \phi}{\partial y} \rangle + \beta_2 z \langle y \frac{\partial \phi}{\partial y} \rangle \right) \\
V^\delta &= \frac{\sqrt{\delta}}{2} \frac{\nu_2 \sqrt{2} \rho_2}{\sqrt{m_2}} (\beta_1 m_1 + \beta_2 z) z \bar{\sigma}'(z),
\end{aligned}
\tag{5.30}
$$

we can write the approximating formula (5.27) as

$$\tilde{C}^{\epsilon,\delta} = C_0 - \left(V^\epsilon x \frac{\partial}{\partial x} - (T-t) V^\delta x \frac{\partial}{\partial x} \right) \frac{\partial C_0}{\partial \sigma}. \tag{5.31}$$

We will see in the next section that the parameters V^ϵ and V^δ can be obtained by calibrating expanded implied volatility function to observed implied volatility surface.

5.3.4   Implied volatility

Recall that the implied volatility I is defined as the volatility value which will equate the Black-Scholed pricing function C_{BS} to the corresponding option price observed on the market:

$$C_{BS}(t, x; T, K, I) = C^{observed}(t, x; T, K).$$

Since we also have

$$C^{\epsilon,\delta} = C^{observed}(t, x; T, K),$$

we have

$$C_{BS}(t, x; T, K, I) = C^{\epsilon,\delta}. \tag{5.32}$$

On the one hand, we can expand the implied volatility as

$$I = I_0 + I_1^{\epsilon} + I_1^{\delta} + \cdots , \tag{5.33}$$

in which I_1^{ϵ} and I_1^{δ} are proportional to $\sqrt{\epsilon}$ and $\sqrt{\delta}$, respectively.

Next, we apply Taylor expansion to the Black-Scholes prcing function $C_{BS}(t, x; T, K, I)$ w.r.t the implied volatility parameter around I_0 and we have:

$$C_{BS}(t, x; T, K, I) = C_{BS}(I_0) + (I_1^{\epsilon} + I_1^{\delta})\frac{\partial C_{BS}}{\partial \sigma}(I_0) + \cdots . \tag{5.34}$$

On the other hand, we have deduced that

$$C^{\epsilon,\delta} \approx C_0 - \left(V^{\epsilon}x\frac{\partial}{\partial x} - (T - t)V^{\delta}x\frac{\partial}{\partial x}\right)\frac{\partial C_0}{\partial \sigma}. \tag{5.35}$$

Matching terms of corresponding orders in (5.34) and (5.35) we have

$$I_0 = \bar{\sigma}(z),$$

$$I_1^\epsilon \frac{\partial C_{BS}}{\partial \sigma}(I_0) = -V^\epsilon x \frac{\partial^2 C_0}{\partial x \partial \sigma}, \tag{5.36}$$

$$I_1^\delta \frac{\partial C_{BS}}{\partial \sigma}(I_0) = (T-t)V^\delta x \frac{\partial^2 C_0}{\partial x \partial \sigma}.$$

Combined with the fact that

$$\left(x\frac{\partial}{\partial x}\right)\frac{\partial C_{BS}}{\partial \sigma} = \left(1 - \frac{d_1}{\sigma\sqrt{T-t}}\right)\frac{\partial C_{BS}}{\partial \sigma}, \tag{5.37}$$

(5.36) implies that

$$I_1^\epsilon = -V^\epsilon\left(1 - \frac{d_1}{\sigma\sqrt{T-t}}\right),$$

$$I_1^\delta = (T-t)V^\delta\left(1 - \frac{d_1}{\sigma\sqrt{T-t}}\right). \tag{5.38}$$

Substituting $I_0 = \bar{\sigma}(z)$ and (5.38) into (5.33), we have that the implied volatility function can be approximated as

$$I \approx \bar{\sigma} - V^\epsilon\left(1 - \frac{d_1}{\sigma\sqrt{T-t}}\right) + (T-t)V^\delta\left(1 - \frac{d_1}{\sigma\sqrt{T-t}}\right). \tag{5.39}$$

Substituting in

$$d_1 = \frac{\ln(x/K) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}},$$

we have

$$I \approx \bar{\sigma} + \frac{V^\epsilon}{\bar{\sigma}} \frac{\ln(x/K)}{T-t} + V^\epsilon \frac{(r + \frac{\bar{\sigma}^2}{2})}{\bar{\sigma}} + V^\delta(T-t)\left(1 - \frac{r + \frac{\bar{\sigma}^2}{2}}{\bar{\sigma}}\right) - \frac{V^\delta}{\bar{\sigma}}\ln(x/K)$$

$$= \bar{\sigma} + V^\epsilon\left(\frac{\ln(x/K)}{\bar{\sigma}(T-t)} + \frac{(r + \frac{\bar{\sigma}^2}{2})}{\bar{\sigma}}\right) + V^\delta(T-t)\left(1 - \frac{r + \frac{\bar{\sigma}^2}{2}}{\bar{\sigma}}\right) - \frac{V^\delta}{\bar{\sigma}}\ln(x/K). \tag{5.40}$$

This shows that the parameters V^ϵ and V^δ can be estimated by calibrating (5.40) can be calibrated to implied volatility surfaces.

Chapter 6

Discussion

The asymptotic expansion methods provide us with a relativly general tool for solving stochastic volatility option pricing models. It exploits the fact that asset returns volatility processes sometimes are fast mean-reverting.

However, several problems with the methods need to be pointed out. First of all, the methods depend heavily on the assumption that volatility is fast mean-reverting, i.e. the rate of mean-reverting of the volatility driving process is large. However, this point was only checked for a sample of high-frequency $S\&P500$ index data. Whether volatilities of different assets are fast mean-reverting in general have not been tested. Thus, the general applicability of the methods may be questioned. Also, high-frequency data are known to be influenced by the *market micro-structure* phenomenon. So whether the fast mean-reverting is a characteristic of volatility processes in general, or is it a characteristic of specific asset (e.g. caused by the liquidity of the asset) also needs to be investigated. And most importantly, the convergence of the asymptotic expansion methods depends wholly on the reciprocal of the rate of mean-reverting. So that if the fast mean-reverting assumption is violated, then

the valitility of the methods will be seriously flawed.

Secondly, as it pointed out by the authors in [7] as well as in our remarks in section 4.3, the asymptotic methods are model-insensitive, meaning that different models will wind up with structurally identical pricing formulas. Notice that parameters in the models can be further estimated after one calibrates V_2 and V_3. It would be an interesting experiment to do if one can make further inference of the parameters in the models using those V_2 and V_3 to see whether those common parameters, e.g. rate of mean-reverting and volatility risk parameter, in different models will have similar values.

Finally, although the asymptotic expansion methods result in implied volatility functions with rich structures and thus fitting ability, the out-of-sample pricing power of the resulting price formulas remains unknown. Actually most of the literature associated with the asymptotic expansion method emphasize how well their model implied volatility function fits observed implied volatility surface, whereas the pricing performance of the resulting formula was almost untouched. One may notice that as long as one incorporates more randomness into the model, one will get richer structure for the model implied volatility function, which will surely fit observed implied volatility surface better. One example that illustrates this point is [3], in which the author built a model with five Brownian motions and wind up with an implied volatility function that fits data almost perfectly, as alledged by the author. But there is no guarantee that a model that fits implied volatility data better will also fit option price better. Actually it had been showed that stochastic volatility option pricing models do not have very good out-of-sample pricing ability (see [1]). Since the asymptotic expansion methods is model-insensitive, it probably inherits features of option pricing formulas of stochastic volatility models in general.

The above points may be further investigated in order to make the asymptotic expansion methods more convincing.

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