Stability properties of a crack inverse problem in half space

Andrew Murdza
Advised by Professor Darko Volkov

A Major Qualifying Project, Presented to the Faculty of Worcester Polytechnic Institute in Partial Fulfillment of the Requirements for the Degree in Bachelor of Sciences by

---

Andrew Murdza
Advised and Approved by

Dr. Darko Volkov

Contents

1 Abstract

We study in this thesis an inverse problem that originates in geophysics. In this inverse problem, a fault and a slip field have to be determined from an overdetermined Partial Differential Equation (PDE) in half space. We achieve three main goals: first, an existence and uniqueness theorem for this PDE in adequate functional spaces. Second, we show that our overdetermined PDE (which reflects that in practice, geophysicists use a PDE model and have access to boundary measurements) does make it possible to recover the fault and the slip. Third, we show that this recovery is stable, under the assumption that the fault must be planar.
Related questions were solved in the case of three dimensional linear elasticity [11, 9, 12] In this thesis we use a PDE model relevant to the important cases of the anti plane strain configuration, and the plane strain configuration.

2 Introduction

This thesis relates to a problem in geophysics where seismic and displacements data are collected by sensors and then processed using partial differential equations (PDE) models and inverse problem formulations. The goal of the mathematical and computational processing of this data is to determine the geometry of faults, total slip between plates, and accumulated mechanical stress. Recently, much work has been done in the case of the three dimensional linear elasticity model. In particular, well posedness of the forward problem and uniqueness for the inverse problem were shown in [11], while a stability result in the case of planar faults was achieved in [9].

In this thesis, we examine the case of a model involving the Laplace equation. This model is also relevant to geophysics: in dimension two, it relates to the so called anti plane strain configuration, while in dimension three, it relates to the plane strain configuration, [7]. The plane strain configuration has already attracted much attention from geophysicists and mathematicians due to the simplicity of the formulation [3, 4]. Despite how simple this formulation is, it still captures important physical features of strike slip seismic events [1]. The main three results of this thesis are, first, the correct functional space formulation of the model leading to a proof of existence and uniqueness for the direct problem. The direct problem is set up in half space with zero Neumann condition on the top plane (this models a no force condition), the Laplace equation in this half space minus a cut, or crack, which we will rather call a fault to emphasize the connection between our work and geophysics. Across the fault we require the normal derivative of the solution to the PDE to be continuous (continuity of forces), and a forcing term: the discontinuity of the solution, which models the slip. Finally, a finite energy condition is imposed through the use of specific functional spaces.

The second result of this thesis concerns the uniqueness of the related inverse problem: we show that if the value of the solution to this PDE is given in a relatively open set of the top boundary (in geophysics this is the measured data), then the PDE becomes overdetermined and the fault and the slip can be inferred.

The third (and most challenging) result that we prove is that the determination of the geometry of this fault from these boundary measurements is actually Lipschitz stable, if it is assumed that the geometry is planar. We plan to submit these results to a peer reviewed journal soon.

We now introduce notations and equations used throughout this thesis. Using the standard rectangular coordinates $x = (x_1, x_2, x_3)$ of $\mathbb{R}^3$, we define $\mathbb{R}^{3-}$ to be the open half space $x_3 < 0$.

Let Γ be a C^2 regular open surface whose closure is contained in $\mathbb{R}^{3-}$ and $g \in H^{1/2}(\Gamma)$. Define a unit normal of Γ (of class C^1) by $\hat{n}$. We will show that the boundary value problem

$$\Delta u = 0 \text{ in } \mathbb{R}^{3-} \setminus \overline{\Gamma}, \tag{1}$$

$$\frac{\partial u}{\partial x_3} = 0 \text{ on the surface } x_3 = 0, \tag{2}$$

$$\frac{\partial u}{\partial \hat{n}} \text{ is continuous across } \Gamma, \tag{3}$$

$$[u] = g \in H^{1/2}(\Gamma) \text{ is a given jump across } \Gamma, \tag{4}$$

$$u = O\left(\frac{1}{|x|^2}\right), \tag{5}$$

has a unique solution. We will prove that Γ (which in some models plays the role of a crack) and the jump g can be uniquely determined by the value of u on any relatively open non-empty subset W of $\{x_3 = 0\}$. We will also derive a Lipchitz estimate for the Hausdorff distance between cracks corresponding to two different sets of input data on W.

3 Preliminary Results: Elementary Differential Geometry and Potential Theory

Definition 3.1. Let V be a $d-1$ dimensional closed surface in $\mathbb{R}^d$ of class C^2. For each local chart (U_α, ϕ_α) in the atlas of V and for each $y \in \phi_\alpha^{-1}(U_\alpha)$ we set

$$dS(y) = \left|\det\left(A(\phi_\alpha(y))\right)\right|^{1/2},$$

where $A(u)$ is a matrix whose ij^{th} entry is $\frac{\partial \phi_\alpha^{-1}}{\partial y_i}(u) \cdot \frac{\partial \phi_\alpha^{-1}}{\partial y_j}(u)$.

For simplicity, all surfaces considered in thesis will be of class C^2.

Definition 3.2. Let D be a connected, and bounded domain of $\mathbb{R}^d$ with boundary ∂D. We define the outward unit normal of ∂D at $y \in \partial D$ to be

$$\hat{n}(y) = \nabla s(y), \tag{6}$$

where

$$s(x) = \begin{cases} -d(x, \partial D) & \text{if } x \in \overline{D} \\ d(x, \partial D) & \text{if } x \in \mathbb{R}^d \setminus \overline{D} \end{cases}$$

and $d(x, \partial D) = \inf_{y \in \partial D} d(x, y)$. We know that s is twice continuously differentiable in a neighborhood of ∂D (see Section 2.5.6 of [6]).

Lemma 3.1. *Let ∂D be the boundary of a connected, bounded, open, and C^2 regular domain D of $\mathbb{R}^d$ and let $\hat{n}$ be the outward unit normal of ∂D. Then there is a positive constant L such that*

$$|\hat{n}(x) \cdot (x - y)| \leq L \, |x - y|^2, \ \forall x, y \in \partial D.$$

Proof. For any $x_0 \in D$, there is an $r > 0$ and $\phi \in C^2(\overline{B(x_0, r)})$ such that $\overline{B(x_0, r)} \cap \partial D = \phi^{-1}(0)$ and $\nabla \phi \neq 0$. Without loss of generality, we can assume that

$$\hat{n} = \frac{\nabla \phi}{|\nabla \phi|}$$

(otherwise replace ϕ by $-\phi$). By Taylor's Theorem, we know that for all $x \in \overline{B(x_0, r)}$ and $h \in \mathbb{R}^d$ such that $x + h \in \overline{B(x_0, r)}$,

$$\phi(x + h) = \phi(x) + \nabla \phi(x) \cdot h + O(|h|^2).$$

If x and $x + h$ are on ∂D, then $\phi(x + h) = \phi(x) = 0$ so

$$\nabla \phi(x) \cdot h = O(|h|^2).$$

Since $\overline{B(x_0, r)}$ is compact, there is $L_{x_0, r} > 0$ depending on x_0 and r such that

$$|\nabla \phi(x)| \geq L_{x_0, r}, \ \forall x \in \overline{B(x_0, r)}.$$

Since ∂D is compact, we can cover ∂D by a finite number N of closed balls L_{x_i, r_i}. Thus

$$|\nabla \phi(x)| \geq L, \ \forall x \in \partial D,$$

where

$$L = \max_{1 \leq i \leq N} L_{x_i, r_i}.$$

Thus

$$\hat{n} \cdot h = \frac{\nabla \phi}{|\nabla \phi|} \cdot h = O(|h|^2)$$

and Lemma 3.1 is proved.

$\blacksquare$

Definition 3.3. Recall that the fundamental solution to the Laplace Equation in $\mathbb{R}^d$ is

$$\Phi(x,y) = \begin{cases} -\frac{1}{2\pi} \ln |x-y| & d = 2 \\ \frac{1}{(d-2)\omega_d |x-y|^{d-2}} & d \geq 3 \end{cases},$$

where ω_d is the surface area of the unit ball in $\mathbb{R}^d$.

Definition 3.4. Let D be an open, connected, and bounded domain. Let ∂D be the boundary of D and $\hat{n}$ be the outward unit normal of ∂D. Let u be defined in a neighborhood of ∂D except possibly on ∂D. For $z \in \partial D$ we define the jump in u across ∂D at z to be

$$[u](z) = \lim_{h \to 0^+} u(z + h\hat{n}(z)) - u(z - h\hat{n}(z)),$$

if this limit exists.

Lemma 3.2. *Let D, and $\hat{n}$ be as in the previous definition and Φ as in definition 3.3. Let ψ be in the Sobolev space $H^{\frac{1}{2}}(\partial D)$. Let*

$$q(x) = \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \hat{n}(y)} \psi(y) dS(y).$$

Then $q \in H^1(D)$ and there is a constant C such that

$$\|q\|_{H^1(D)} \leq C \|\psi\|_{H^{1/2}(\partial D)}. \tag{7}$$

We skip the proof of lemma 3.2. This proof can be built based on properties of the double layer potential for continuous densities shown in appendix A, Sobolev continuity properties of the surface operator defined by the double layer potential $\frac{\partial \Phi(x,y)}{\partial \hat{n}(y)}$, and elliptic regularity for PDEs.

Definition 3.5. Let Φ, D, ∂D, and $\hat{n}$ be as above and let Γ be an open surface included in

∂D. We define $\tilde{H}^{1/2}(\Gamma)$ to be the set of restrictions to Γ of functions in $H^{1/2}(\Gamma)$ supported in Γ.

Lemma 3.3. *Let D, ∂D, $\hat{n}$, and Φ be as above. Let Γ be an open surface included in ∂D and let $\psi \in \tilde{H}^{1/2}(\Gamma)$. Assume $d = 3$. Then*

$$q(x) = \int_\Gamma \frac{\partial \Phi(x,y)}{\partial \hat{n}(y)} \psi(y) dS(y)$$

is in $H^1(\mathbb{R}^d \setminus \overline{\Gamma})$ and there is a constant C such that

$$\|q\|_{H^1(\mathbb{R}^d \setminus \overline{\Gamma})} \leq C \|\psi\|_{H^{1/2}(\partial D)}.$$

Proof. This can be proved from lemma 3.2 and using the decay at infinity of Φ. ∎

Lemma 3.4. *Let Φ, D, ∂D, and $\hat{n}$ be defined as in Lemma A.3 and $\psi \in H^{1/2}(\partial D)$. Then the jump of*

$$u(x) = \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \hat{n}(y)} \psi(y) dS(y),$$

in the sense of the trace theorem, is equal to $\psi(x)$ almost everywhere.

Proof. From Section 2.5.7 of [6] we know that $u \in H^{3/2}\left(\mathbb{R}^d \setminus \partial D\right)$. Thus the inner and outer traces of u exist by the Trace Theorem. If ψ is in $C^1(\partial D)$ we know from appendix A that the inner and outer traces are

$$\int_{\partial D} \frac{\partial \Phi(z,y)}{\partial \hat{n}(y)} \psi(y) dS(y) - \frac{1}{2}\psi(z) \tag{8}$$

and

$$\int_{\partial D} \frac{\partial \Phi(z,y)}{\partial \hat{n}(y)} \psi(y) dS(y) + \frac{1}{2}\psi(z) \tag{9}$$

respectively, where z is the projection of x onto ∂D. We conclude that

$$[q](z) = \left(\int_{\partial D} \frac{\partial \Phi(z,y)}{\partial \hat{n}(y)} \psi(y) dS(y) + \frac{1}{2}\psi(z) \right) - \left(\int_{\partial D} \frac{\partial \Phi(z,y)}{\partial \hat{n}(y)} \psi(y) dS(y) - \frac{1}{2}\psi(z) \right) = \psi(z). \tag{10}$$

The result follows from density of $C^1(\partial D)$ in $H^{1/2}(\partial D)$. ∎

Lemma 3.5. *Let Φ, D, ∂D, and $\hat{n}$ be defined as in Lemma A.3 and let Γ be an open surface* *included in ∂D. Let $\psi \in \tilde{H}^{1/2}(\Gamma)$. Then the jump of*

$$q(x) = \int_\Gamma \frac{\partial \Phi(x,y)}{\partial \hat{n}(y)} \psi(y) dS(y)$$

is equal to $\psi(x)$ almost everywhere.

Proof. This lemma follows from Lemma 3.4. ■

Lemma 3.6. *Let $u \in H^1_{loc}(\mathbb{R}^{3-} \setminus \overline{\Gamma})$. If $[u] = 0$ across Γ, then $u \in H^1_{loc}(\mathbb{R}^{3-})$.*

Proof. Suppose that u is in $H^1_{loc}(\mathbb{R}^{3-} \setminus \overline{\Gamma})$ such that $[u] = 0$. We know that $\lim_{h \to 0^+} u(z + h\hat{n}(z))$ and $\lim_{h \to 0^-} u(z + h\hat{n}(z))$ exist in the sense of traces because $u \in H^1_{loc}(\mathbb{R}^{3-} \setminus \overline{\Gamma})$. As $[u] = 0$, these limits are the same and $\lim_{h \to 0} u(z + h\hat{n}(z))$ exists $\forall z \in \Gamma$. We may extend u from $\mathbb{R}^{3-} \setminus \overline{\Gamma}$ to $\mathbb{R}^{3-}$ by setting $u(z) = \lim_{h \to 0} u(z + h\hat{n}(z))$ for $z \in \Gamma$. Let Ω be a bounded open subset of $\mathbb{R}^{3-}$. We may extend Γ to a surface Γ' so that there are non-empty subsets Ω^+ and Ω^- of Ω such that $\overline{\Omega} = \overline{\Omega^+} \cup \overline{\Omega^-}$, $\Omega^+ \cap \Omega^- = \emptyset$, and $\partial\Omega^+ \cap \partial\Omega^- = \overline{\Gamma'}$ (see Figure 3). Since $u \in H^1_{loc}(\mathbb{R}^{3-} \setminus \overline{\Gamma})$, ∇u is defined on Ω^- and Ω^+. Let $\hat{n}_+$ and $\hat{n}_-$ be the unit normals of Γ which point towards Ω^- and towards Ω^+ respectively. Let the outward unit normal of Γ be defined as pointing towards Ω^-. Then for $\psi \in C_c^\infty(\Omega)$ we have

$$\int_\Omega u \frac{\partial \psi}{\partial x_i} dx = \int_{\Omega^+} \nabla \cdot (u\psi e_i) dx + \int_{\Omega^-} \nabla \cdot (u\psi e_i) dx - \int_\Omega \frac{\partial u}{\partial x_i} \psi dx.$$

We apply the Divergence Theorem to conclude that

$$\int_\Omega u \frac{\partial \psi}{\partial x_i} dx = \int_{\partial\Omega^+} (u\psi e_i) \cdot \hat{n}_+ dS + \int_{\partial\Omega^-} (u\psi e_i) \cdot \hat{n}_- dS - \int_\Omega \frac{\partial u}{\partial x_i} \psi dx$$

Since $\psi = 0$ on $\partial\Omega$, we see that

$$\int_\Omega u \frac{\partial \psi}{\partial x_i} dx = \int_{\Gamma'} (u\psi e_i) \cdot \hat{n}_+ dS + \int_{\Gamma'} (u\psi e_i) \cdot \hat{n}_- dS - \int_\Omega \frac{\partial u}{\partial x_i} \psi dx$$

Since $\hat{n}_+$ and $\hat{n}_-$ point in opposite directions, we find that $\hat{n}_+ = -\hat{n}_-$ and

$$\int_\Omega u \frac{\partial \psi}{\partial x_i} dx = -\int_\Omega \frac{\partial u}{\partial x_i} \psi dx.$$

We see that u has a weak first order-derivatives defined on Ω. We conclude that $u \in H^1_{loc}(\mathbb{R}^{3-})$ as required. ■

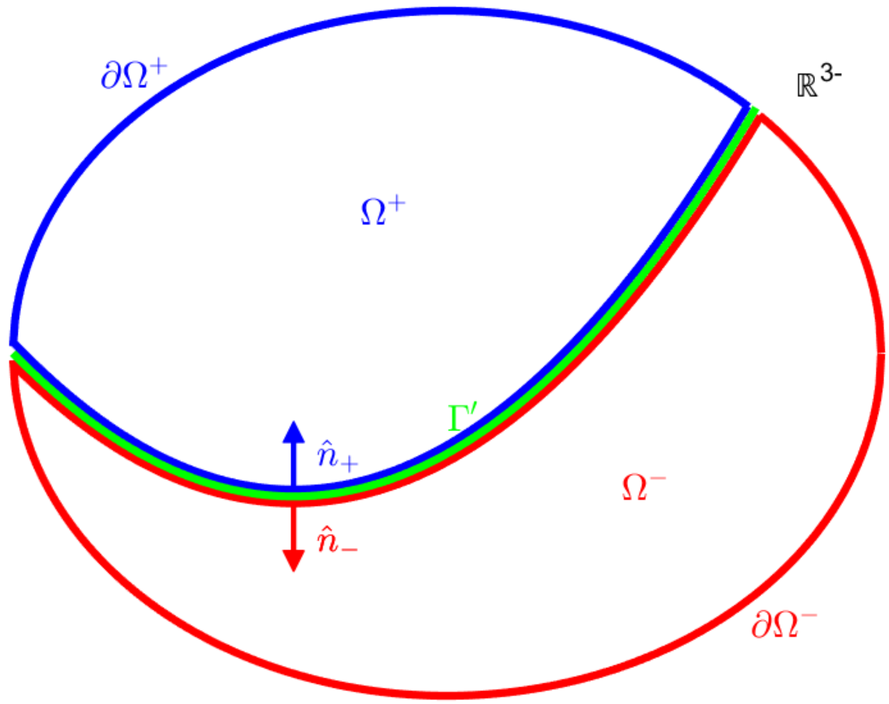


Figure 1: A graph of the curve Γ after it has been extended to divide the domain Ω into two smaller domains.

4 Norm Equivalence Lemmas

Lemma 4.1 (Hardy's Inequality). *Let $f \in C_c^\infty(\mathbb{R}^3)$. Then there is a constant $C > 0$ such that*

$$\int_{\mathbb{R}^3} \frac{f^2}{|x|^2} \leq 4 \int_{\mathbb{R}^3} |\nabla f|^2 . \tag{11}$$

Proof. First suppose that $f \in C_c^\infty(\mathbb{R}^3)$ is radial. Then f is supported in $B(0, R)$ for some $R > 0$. We know that

$$\begin{aligned}
\int_{\mathbb{R}^3} \frac{f(x)^2}{|x|^2} dx &= \int_{B(0,R)} \frac{f(x)^2}{|x|^2} dx \\
&= 4\pi \int_0^A f(\rho)^2 d\rho \\
&= -4\pi \int_0^A \rho 2 f(\rho) f'(\rho) d\rho \\
&\leq 2 \left(4\pi \int_0^A \frac{f(\rho)^2}{\rho^2} \rho^2 d\rho \right)^{1/2} \left(4\pi \int_0^A f'(\rho)^2 \rho^2 d\rho \right)^{1/2} \\
&= 2 \left(\int_{\mathbb{R}^3} \frac{f(x)^2}{|x|^2} dx \right)^{1/2} \left(\int_{\mathbb{R}^3} |\nabla f(x)|^2 dx \right)^{1/2}
\end{aligned}$$

We conclude that

$$\int_{\mathbb{R}^3} f(x)^2 dx \le 4 \int_{\mathbb{R}^3} |\nabla f(x)|^2 \, dx.$$

If f is not radial, the proof is about the same after a conversion to spherical coordinates. ∎

Lemma 4.2. *Let $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\frac{f}{(1+|x|^2)^{1/2}}$ and ∇f are in $L^2(\mathbb{R}^3)$. Then*

$$\int_{\mathbb{R}^3} \frac{|f|^2}{1+|x|^2} \le 4 \int_{\mathbb{R}^3} |\nabla f|^2 .$$

Proof. Let $h \in C^\infty(\mathbb{R}^3)$ such that $\frac{h}{(1+|x|^2)^{1/2}}$ and ∇h are in $L^2(\mathbb{R}^3)$. Let $p \in C_c^\infty(\mathbb{R}^3)$ such that $0 \le p \le 1$, $p(x) = 1$ for $|x| \le 1$, and $p(x) = 0$ for $|x| \ge 2$. Set $p_n(x) = p\left(\frac{x}{n}\right)$. Clearly $hp_n \in C_c^\infty(\mathbb{R}^3)$, so we may apply (11) to find that

$$\int_{\mathbb{R}^3} \frac{|hp_n|^2}{|x|^2} \le 4 \int_{\mathbb{R}^3} |\nabla (hp_n)|^2 .$$

In particular, we know that

$$\int_{\mathbb{R}^3} \frac{|hp_n|^2}{1+|x|^2} \le 4 \int_{\mathbb{R}^3} |\nabla (hp_n)|^2 . \tag{12}$$

Clearly $p_n \to 1$, so $hp_n \to h$ pointwise. We also know that

$$\frac{|hp_n|^2}{1+|x|^2} \le \frac{|h|^2}{1+|x|^2} \in L^1(\mathbb{R}^3)$$

since $0 \le p_n \le 1$ and $\frac{h}{(1+|x|^2)^{1/2}} \in L^2(\mathbb{R}^3)$. Thus

$$\int_{\mathbb{R}^3} \frac{|hp_n|^2}{1+|x|^2} \to \int_{\mathbb{R}^3} \frac{|h|^2}{1+|x|^2} \tag{13}$$

by the Dominated Convergence Theorem. We know that

$$\nabla (h(x)p_n(x)) = (\nabla h(x))p_n(x) + \frac{1}{n}(\nabla p)\left(\frac{x}{n}\right) h(x) \to \nabla h(x)$$

since $p_n \to 1$ and $\nabla p \in C_c^\infty(\mathbb{R}^3)$ is bounded. As p is constant outside of $\{1 \le |x| \le 2\}$, $\nabla p_n = 0$ outside of $\{n \le |x| \le 2n\}$ and

$$|\nabla (hp_n)| \le |\nabla h| + \frac{1}{n} \max_{\mathbb{R}^3} |\nabla p| I_{n \le |x| \le 2n} |h| .$$

Since $\frac{1}{n} \leq \frac{2}{|x|}$, $\forall x \in \{n \leq |x| \leq 2n\}$, we have

$$\begin{aligned} |\nabla(hp_n)| &\leq |\nabla h| + 2\frac{1}{|x|}\max_{\mathbb{R}^3}|\nabla p| I_{n\leq|x|\leq 2n}\,|h| \\ &= |\nabla h| + 2\frac{|h|}{|x|}\max_{\mathbb{R}^3}|\nabla p| I_{|x|\geq 1} \\ &\leq |\nabla h| + 2(2)^{1/2}\frac{|h|}{(1+|x|^2)^{1/2}}\max_{\mathbb{R}^3}|\nabla p|. \end{aligned}$$

The first term is in $L^2(\mathbb{R}^3)$ since we assumed $\nabla h \in L^2(\mathbb{R}^3)$ and second term is in $L^2(\mathbb{R}^3)$ because $\frac{|h|}{(1+|x|^2)^{1/2}} \in L^2(\mathbb{R}^3)$. Thus $|\nabla(hp_n)|^2$ is dominated by a $L^1(\mathbb{R}^3)$ function. We also know that $\nabla(hp_n) \to \nabla h$, so

$$\int_{\mathbb{R}^3} |\nabla(hp_n)|^2 \to \int_{\mathbb{R}^3} |\nabla h|^2 \tag{14}$$

by the Dominated Convergence Theorem. Letting $n \to \infty$ in (12), we use (13)-(14) to conclude that

$$\int_{\mathbb{R}^3} \frac{|h|^2}{1+|x|^2} \leq 4\int_{\mathbb{R}^3} |\nabla h|^2 . \tag{15}$$

Let $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\frac{f}{(1+|x|^2)^{1/2}}, \nabla f \in L^2(\mathbb{R}^3)$. Set $\rho_n = C_n p\,(nx)$, where the constants C_n are chosen so that $\int_{\mathbb{R}^3} \rho_n = 1$. Then $\rho_n \star f \in C^\infty(\mathbb{R}^3)$ and (15) implies that

$$\int_{\mathbb{R}^3} \frac{|\rho_n \star f|^2}{1+|x|^2} \leq 4\int_{\mathbb{R}^3} |\nabla(\rho_n \star f)|^2 \tag{16}$$

We know that $\rho_n \star f$ converges to f in $L^2(\mathbb{R}^3)$ so

$$\left\| \frac{f}{(1+|x|^2)^{1/2}} - \frac{\rho_n \star f}{(1+|x|^2)^{1/2}} \right\|_2 \leq \|f - \rho_n \star f\|_2 \left\| \frac{1}{(1+|x|^2)^{1/2}} \right\|_2 \to 0$$

Thus $\frac{\rho_n \star f}{(1+|x|^2)^{1/2}} \to \frac{f}{(1+|x|^2)^{1/2}}$ in $L^2(\mathbb{R}^3)$. So $\left\|\frac{\rho_n \star f}{(1+|x|^2)^{1/2}}\right\|_2 \to \left\|\frac{f}{(1+|x|^2)^{1/2}}\right\|_2$, i.e.,

$$\int_{\mathbb{R}^3} \frac{|\rho_n \star f|^2}{1+|x|^2} \to \int_{\mathbb{R}^3} \frac{|f|^2}{1+|x|^2}. \tag{17}$$

As $\nabla(\rho_n \star f) = \rho_n \star \nabla f \to \nabla f$ in $L^2(\mathbb{R}^3)$, $\|\nabla(\rho_n \star f)\| \to \|\nabla f\|_2$, i.e.,

$$\int_{\mathbb{R}^3} |\nabla(\rho_n \star f)|^2 \to \int_{\mathbb{R}^3} |\nabla f|^2 . \tag{18}$$

Letting $n \to \infty$ in (16), and making use of (17)-(18), we conclude that

$$\int_{\mathbb{R}^3} \frac{|f|^2}{1+|x|^2} \leq 4 \int_{\mathbb{R}^3} |\nabla f|^2$$

as required. ■

Lemma 4.3. *Let $\mathcal{V}(\mathbb{R}^{3-})$ be the space of functions u defined on $\mathbb{R}^{3-}$ such that ∇u and $\frac{u}{(1+|x|^2)^{1/2}}$ are in $L^2(\mathbb{R}^{3-})$. Then the following norms are equivalent on $\mathcal{V}(\mathbb{R}^{3-})$*

$$\|u\|_{1,\mathbb{R}^{3-}} = \left(\int_{\mathbb{R}^{3-}} |\nabla u|^2\right)^{1/2} + \left(\int_{\mathbb{R}^{3-}} \frac{|u|^2}{1+|x|^2}\right)^{1/2},$$

$$\|u\|_{2,\mathbb{R}^{3-}} = \left(\int_{\mathbb{R}^{3-}} |\nabla u|^2\right)^{1/2}.$$

Proof. Clearly $\|u\|_{2,\mathbb{R}^{3-}} \leq \|u\|_{1,\mathbb{R}^{3-}}$, $\forall u \in \mathcal{V}(\mathbb{R}^{3-})$
To show that $\|u\|_{1,\mathbb{R}^{3-}}$ and $\|u\|_{2,\mathbb{R}^{3-}}$ are equivalent, we must find $C > 0$ such that

$$\|u\|_{1,\mathbb{R}^{3-}} \leq C \|u\|_{2,\mathbb{R}^{3-}}, \ \forall u \in \mathcal{V}(\mathbb{R}^{3-})$$

Let $u \in \mathcal{V}(\mathbb{R}^{3-})$ and define $\overline{u} : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$\overline{u} = \begin{cases} u(x_1, x_2, x_3) & \text{if } x_3 \leq 0 \\ u(x_1, x_2, -x_3) & \text{if } x_3 > 0. \end{cases}$$

Then $[\overline{u}] = 0$ across $\{x_3 = 0\}$, so $\overline{u} \in H^1(\mathbb{R}^3)$ by Lemma 3.6. As $\frac{|u|^2}{1+|x|^2} \in L^2(\mathbb{R}^{3-})$, we know that

$$\int_{\mathbb{R}^3} \frac{|\overline{u}|^2}{1+|x|^2} = \int_{\mathbb{R}^{3-}} \frac{|\overline{u}|^2}{1+|x|^2} + \int_{\mathbb{R}^{3+}} \frac{|\overline{u}|^2}{1+|x|^2} = 2\int_{\mathbb{R}^{3-}} \frac{|u|^2}{1+|x|^2}. \tag{19}$$

Similarly,

$$\int_{\mathbb{R}^3} |\nabla \overline{u}|^2 = 2\int_{\mathbb{R}^{3-}} |\nabla u|^2. \tag{20}$$

Applying Lemma 4.2 to $\overline{u}$, we find that

$$\int_{\mathbb{R}^3} \frac{\overline{u}_i}{1+|x|^2} \leq 4 \int_{\mathbb{R}^3} |\nabla \overline{u}_i|^2 \tag{21}$$

Substituting (19)-(20) into (21), we get

$$2\int_{\mathbb{R}^{3-}} \frac{u}{1+|x|^2} \leq 4\left(2\int_{\mathbb{R}^{3-}} |\nabla u|^2\right),$$

Thus

$$\int_{\mathbb{R}^{3-}} \frac{u}{1+|x|^2} + \int_{\mathbb{R}^{3-}} |\nabla u|^2 \leq 5\int_{\mathbb{R}^{3-}} |\nabla u|^2 .$$

We conclude that $\|u\|_{1,\mathbb{R}^{3-}} \leq 5\,\|u\|_{2,\mathbb{R}^{3-}}$ and the norms $\|\cdot\|_{1,\mathbb{R}^{3-}}$ and $\|\cdot\|_{2,\mathbb{R}^{3-}}$ are equivalent.

■

Lemma 4.4. *Let Γ be a C^2 regular open surface whose closure is contained in $\mathbb{R}^{3-}$. Let $\mathcal{V}$ be the vector space of scalar functions u defined in $\mathbb{R}^{3-}\setminus\overline{\Gamma}$ such that ∇u and $\frac{u}{(1+r^2)^{1/2}}$ are in $L^2(\mathbb{R}^{3-}\setminus\overline{\Gamma})$. Then the following two norms are equivalent on $\mathcal{V}$*

$$\|u\|_1 = \left(\int_{\mathbb{R}^{3-}\setminus\overline{\Gamma}} |\nabla u|^2\right)^{1/2} + \left(\int_{\mathbb{R}^{3-}\setminus\overline{\Gamma}} \frac{|u|^2}{1+|x|^2}\right)^{1/2}$$

$$\|u\|_2 = \left(\int_{\mathbb{R}^{3-}\setminus\overline{\Gamma}} |\nabla u|^2\right)^{1/2}.$$

Proof. Clearly $\|u\|_2 \leq \|u\|_1$, so it is sufficient to show that $\exists C>0, \|u\|_1 \leq C\,\|u\|_2\,,\ \forall u\in\mathcal{V}$.
Arguing by contradiction, suppose that $\forall C>0$, $\exists u\in\mathcal{V}$ such that $\|u\|_1 > C\,\|u\|_2$.
Then $\forall n\in\mathbb{N}$, $\exists u_n\in\mathcal{V}$ such that $\|u_n\|_1 > n\,\|u_n\|_2$, i.e.,

$$\int_{\mathbb{R}^{3-}\setminus\overline{\Gamma}} |\nabla u_n|^2 + \int_{\mathbb{R}^{3}\setminus\overline{\Gamma}} \frac{|u_n|^2}{1+|x|^2} > n\int_{\mathbb{R}^{3-}\setminus\overline{\Gamma}} |\nabla u_n|^2 \tag{22}$$

We may assume that

$$\int_{\mathbb{R}^{3-}\setminus\overline{\Gamma}} \frac{|u_n|^2}{1+|x|^2} = 1 \tag{23}$$

(otherwise divide u_n by $\int_{\mathbb{R}^{3-}\setminus\overline{\Gamma}} \frac{|u_n|^2}{1+|x|^2}$).
Let D be an open set whose boundary ∂D is regular and contains Γ. Define

$$q_n(x) = \int_{\partial D} \frac{\partial\Phi(x,y)}{\partial\hat{n}(y)}[u_n(y)], \qquad x\in\mathbb{R}^{3-}\setminus\overline{\Gamma}.$$

Let Ω be a bounded open set containing Γ whose closure is included in $\mathbb{R}^{3-}$.

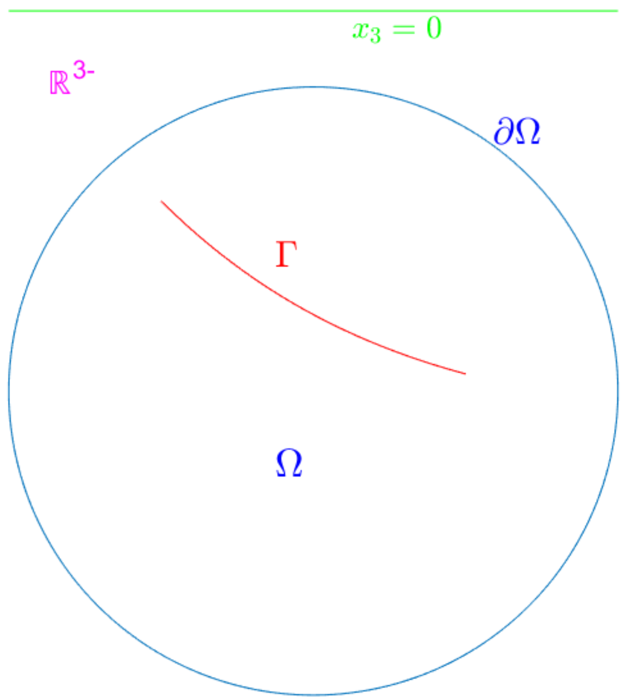


Figure 2: A sketch of the domain Ω.

Equation (22) implies that

$$\int_{\mathbb{R}^{3-}\setminus\overline{\Gamma}} |\nabla u_n|^2 < \frac{1}{n-1}. \tag{24}$$

Thus

$$\|\nabla u_n\|_{L^2(\Omega\setminus\overline{\Gamma})} \le \|\nabla u_n\|_{L^2(\mathbb{R}^{3-}\setminus\overline{\Gamma})} < 1, \ \forall n \ge 2. \tag{25}$$

As Ω is bounded, $\exists M \in \mathbb{R}$ such that $|x| \le M$, $\forall x \in \Omega$. By (23) we have

$$\begin{aligned} \|u_n\|^2_{L^2(\Omega\setminus\overline{\Gamma})} &\le \int_{\Omega\setminus\overline{\Gamma}} \frac{u_n(x)^2}{1+|x|^2}\left(1+M^2\right) dx \\ &= 1 + M^2. \end{aligned} \tag{26}$$

From (25)-(26) we see that

$$\|u_n\|_{H^1(\Omega\setminus\overline{\Gamma})} = \|u_n\|_{L^2(\Omega\setminus\overline{\Gamma})} + \|\nabla u_n\|_{L^2(\Omega\setminus\overline{\Gamma})} < (1+M^2) + 1.$$

Thus $u_n \in H^1_{loc}(\mathbb{R}^{3-} \setminus \overline{\Gamma})$ and u_n is bounded in $H^1(\Omega \setminus \overline{\Gamma})$. Hence u_n has a subsequence u_{n_k} which converges weakly in $H^1(\Omega \setminus \overline{\Gamma})$ and strongly in $L^2(\Omega \setminus \overline{\Gamma})$ by Lemma B.1 and the Kondrachov embedding theorem. Denote the limit of u_{n_k} by u. Equation (24) implies that $\nabla u_{n_k} \to 0$ in $L^2(\Omega \setminus \overline{\Gamma})$ and $\nabla u = 0$. Since u_{n_k} converges strongly to u in $L^2(\mathbb{R}^{3-})$ and ∇u_{n_k}

converges strongly to $0 = \nabla u$, we find that u_{n_k} converges strongly to u in $H^1(\Omega \setminus \overline{\Gamma})$. As $\nabla u = 0$ on $\Omega \setminus \overline{\Gamma}$, u is constant on $\Omega \setminus \overline{\Gamma}$ by Lemma B.2.

Hence u_{n_k} converges strongly to a constant function u in $H^1(\Omega \setminus \overline{\Gamma})$. By the Trace Theorem, $[u_{n_k}] \to [u] = 0$ in $H^{1/2}(\partial D)$. Because $[u_{n_k}] \to 0$ in $H^{1/2}(\partial D)$, Lemma 3.3 implies that ∇q_{n_k} and $\frac{q_{n_k}}{(1+|x|^2)^{1/2}}$ converge to 0 in $L^2(\mathbb{R}^{3-} \setminus \overline{\Gamma})$. Thus $q_{n_k} \to 0$ in $\mathcal{V}$. Since $u_n \in \mathcal{V}$ and $q_{n_k} \in \mathcal{V}$, $u_{n_k} - q_{n_k} \in \mathcal{V}$. We know that $u_{n_k} - q_{n_k} \in H^1(\Omega \setminus \overline{\Gamma})$ by Lemma 3.3. As Lemma 3.4 implies that $[q_n] = [u_n]$ across Γ, $[u_{n_k} - q_{n_k}] = 0$ and $u_{n_k} - q_{n_k} \in H^1(\Omega)$ by Lemma 3.6. Since $\mathbb{R}^{3-} \setminus \overline{\Gamma}$ contains $\mathbb{R}^{3-} \setminus \Omega$, $\nabla(u_{n_k} - q_{n_k}) \in L^2(\mathbb{R}^{3-} \setminus \Omega)$. Hence $\nabla(u_{n_k} - q_{n_k})$ is in $L^2(\mathbb{R}^{3-})$. We know that $\nabla q_{n_k} \to 0$ and $\nabla u_{n_k} \to 0$ in $L^2(\mathbb{R}^{3-} \setminus \overline{\Gamma})$ so $\|u_{n_k} - q_{n_k}\|_{2,\mathbb{R}^{3-}} = \|\nabla(u_{n_k} - q_{n_k})\|_{L^2(\mathbb{R}^{3-})} \to 0$. Thus $\|u_{n_k} - q_{n_k}\|_{1,\mathbb{R}^{3-}} \to 0$ since $\|\cdot\|_{1,\mathbb{R}^{3-}}$ and $\|\cdot\|_{2,\mathbb{R}^{3-}}$ are equivalent by Lemma 4.3. We also know that $\left\| \frac{q_{n_k}}{(1+|x|^2)^{1/2}} \right\|_{L^2(\mathbb{R}^{3-})} \to 0$ since $q_{n_k} \to 0$ in $\mathcal{V}$. This implies that

$$\int_{\mathbb{R}^{3-} \overline{\Gamma}} \frac{|u_{n_k}|^2}{1+|x|^2} \leq \int_{\mathbb{R}^{3-}} \frac{|u_{n_k} - q_{n_k}|^2}{1+|x|^2} + \int_{\mathbb{R}^{3-}} \frac{|0 - q_{n_k}|^2}{1+|x|^2} \to 0$$

which contradicts (23). We conclude that the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent. ■

5 Existence and Uniqueness Result

Let $G(x, y, \hat{n})$ be the free space Green's function for the Laplace equation such that for any open C^2 surface Γ in $\mathbb{R}^3$ and any displacement u satisfying

$$\Delta u = 0 \text{ in } \mathbb{R}^3 \setminus \Gamma, \tag{27}$$

$$\frac{\partial u}{\partial \hat{n}} \text{ is continuous across } \Gamma, \tag{28}$$

$$[u] = g \in H^{1/2}(\Gamma) \text{ is a given jump across } \Gamma, \tag{29}$$

$$u = O\left(\frac{1}{|x|^2}\right) \text{ and } \nabla u = O\left(\frac{1}{|x|^3}\right) \text{ uniformly as } |x| \to \infty, \tag{30}$$

we have

$$u(x) = \int_\Gamma G(x, y, \hat{n}) g(y), \ \forall x \in \mathbb{R}^3 \setminus \Gamma.$$

We know that $G(x, y, \hat{n}) = \frac{\partial \Phi(x,y)}{\partial \hat{n}(y)}$. We also know that the half space Green's Tensor is $H(x, y, \hat{n}) = \frac{\partial \Phi(x,y)}{\partial \hat{n}(y)} + \frac{\partial \Phi(\overline{x},y)}{\partial \hat{n}}$, where $\overline{x} = (x_1, x_2, -x_3)$. H satisfies (27) through (30) for $x \neq y$

in $\mathbb{R}^{3-}$, and in addition we have that

$$\frac{\partial}{\partial x_3}H(x,y,\hat{n})=0,$$

at $x_3=0$

Theorem 5.1. *The problem* (1)-(4) *has a unique solution in* $\mathcal{V}$ *for* $g\in\tilde{H}^{1/2}(\Gamma)$. *This solution satisfies the decay condition* (5).

Proof. Define

$$u_g(x)=\int_{\partial D}\frac{\partial\Phi(x,y)}{\partial\hat{n}(y)}g(y)dS(y),\ x\in\mathbb{R}^{3-}\setminus\overline{\Gamma},$$

Clearly $u_g\in C^\infty(\mathbb{R}^{3-}\setminus\overline{\Gamma})$. From Lemma 3.4 and Lemma 3.3 we know that $[u_g]=g$ across Γ and $u_g\in H^1(\mathbb{R}^{3-}\setminus\overline{\Gamma})$. Let $\mathcal{U}$ to be the vector space of functions $u\in\mathcal{V}$ such that $[u]=0$ across Γ. Define the bilinear form

$$B(u,v)=\int_{\mathbb{R}^{3-}}(\nabla u\cdot\nabla v)$$

on $\mathcal{V}\times\mathcal{V}$. By Lemma 4.4, $\mathcal{V}$ is a Hilbert space under the norm $\|\cdot\|_2=B(\cdot,\cdot)^{1/2}$ since $\mathcal{V}$ is a Hilbert space under the norm $\|\cdot\|_1$ and $B(\cdot,\cdot)=\|\cdot\|_2$ is equivalent to $\|\cdot\|_1$. As $\mathcal{U}$ is a closed subspace of $\mathcal{V}$, $\mathcal{U}$ is also a Hilbert space under the norm $\|\cdot\|_2=B(\cdot,\cdot)^{1/2}$. Clearly $B(u,v)$ is bilinear, and, since $\|\cdot\|_2=B(\cdot,\cdot)^{1/2}$ defines the default norm on H, B is coercive and continuous. Thus, by the Lax-Milgram Theorem, we know that $\exists!u_0\in\mathcal{U}$ such that $B(u_0,v)=-B(u_g,v),\ \forall v\in\mathcal{U}$. Hence $B(u_0+u_g,v)=B(u_0,v)+B(u_g,v)=0,\ \forall v\in\mathcal{U}$. Set $u=u_0+u_g$. For $v\in C_c^\infty(\mathbb{R}^{3-}\setminus\overline{\Gamma})\subset\mathcal{U}$, we have

$$\int_{\mathbb{R}^{3-}\setminus\overline{\Gamma}}\nabla u\cdot\nabla v dx=0,\tag{31}$$

so $\Delta u=0$ in $\mathbb{R}^{3-}\setminus\overline{\Gamma}$. Now assume that $v\in C_c^\infty(\mathbb{R}^3)$ and v is zero in a neighborhood of $\overline{\Gamma}$. By (31),

$$\int_{\{x_3=0\}}\frac{\partial u}{\partial x_3}v=0.$$

Thus $\frac{\partial u}{\partial x_3}=0$ on $\{x_3=0\}$. As $[u_0]=0$ across Γ, $[u_g]=g$, and $u=u_0+g$, it follows that $[u]=g$ across Γ. We know that $u\in\mathcal{V}$ by construction, so $\frac{u}{\left(1+|x|^2\right)^{1/2}}$ and ∇u are in $L^2(\mathbb{R}^{3-})$.

For $y = (y_1, y_2, y_3)$ in $\mathbb{R}^{3-}$ set $\overline{y} = (y_1, y_2, -y_3)$. Next set

$$w(x) = \int_\Gamma \left[\frac{\partial \Phi}{\partial \hat{n}(y)}(x, y) + \frac{\partial \Phi}{\partial \hat{n}(y)}(x, \overline{y}) \right] g dS(y), \ x \in \mathbb{R}^{3-} \setminus \overline{\Gamma}.$$

It is clear that w is in $\mathcal{V}$ and satisfies (1)-(4). By the uniqueness of the problem (1)-(4) in $\mathcal{V}$, $u = w$. Given that

$$\frac{\partial \Phi}{\partial \hat{n}(y)}(x, y) + \frac{\partial \Phi}{\partial \hat{n}(y)}(x, \overline{y}) = O\left(\frac{1}{|x|^2}\right)$$

and

$$\nabla_x \left[\frac{\partial \Phi}{\partial \hat{n}(y)}(x, y) + \frac{\partial \Phi}{\partial \hat{n}(y)}(x, \overline{y}) \right] = O\left(\frac{1}{|x|^2}\right)$$

uniformly in $\frac{x}{|x|}$, we can claim that $u = O\left(\frac{1}{|x|^2}\right)$ and $\nabla u = O\left(\frac{1}{|x|^3}\right)$ uniformly in $\frac{x}{|x|}$.

∎

6 Inverse Problem Result

Lemma 6.1. *Let U be a open subset of $\mathbb{R}^3$ and $U_- = \{x \in U \,|\, x_3 < 0\}$. Then the only weak solution to the Cauchy problem*

$$\Delta u = 0 \qquad x \in U_-, \tag{32}$$

$$u = 0 \qquad (x_1, x_2, 0) \in \partial U, \tag{33}$$

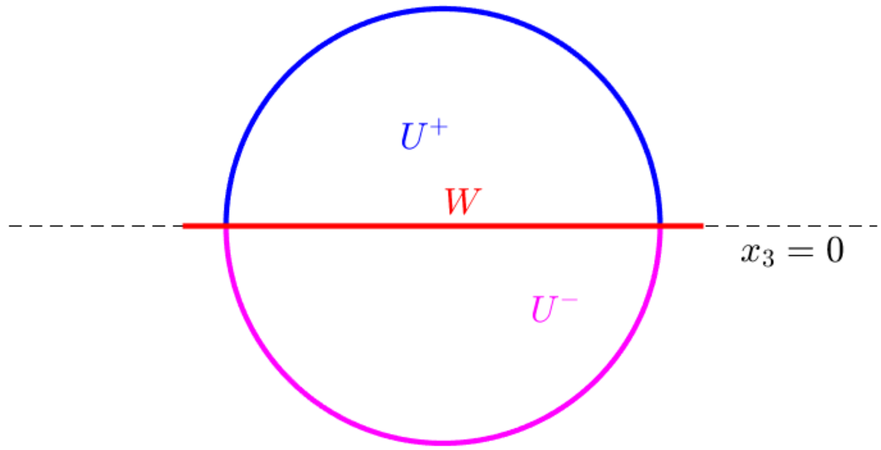
$$\frac{\partial u}{\partial x_3} = 0 \qquad (x_1, x_2, 0) \in \partial U, \tag{34}$$

is $u = 0$.

Proof. Clearly $u = 0$ is a solution to the Cauchy problem. Set $U_+ = \{x \in U | x_3 > 0\}$. We may extend u from U_- to $U = U_+ \cup U_-$ by 0. We will call this extension v. We know that $v \in H^1_{loc}(U)$ and (32) and $v \in C^1(U_-)$ by elliptic regularity. We also know that $\Delta v = 0$ weakly in U by Lemma 3.6. We apply Holmgren's Theorem as stated in Section 2.11 of [linear pde] to find that $u = v = 0$ in U. In particular, we conclude that $v = 0$ in U_-. ∎

Theorem 6.1. *Let Γ_1 and Γ_2 be two bounded open surfaces with smooth boundaries whose closure is contained in $\mathbb{R}^{3-}$. For $i = 1, 2$, assume that u_i solves (1)-(5) with Γ_i in place of Γ and the jumps $g_i \in H^{1/2}(\Gamma)$ in place of g. Let W be a non-empty relatively open subset of $\{x_3 = 0\}$. If $u_1 = u_2$ on W, then $\Gamma_1 = \Gamma_2$ and $g_1 = g_2$.*

Proof. Suppose that $u_1 = u_2$ on W and set $u = u_1 - u_2$. Let U be an open ball centered at the center of W which is small enough that $U \cap \{x_3 = 0\} \subset W$ and $U \cap \overline{\Gamma_1 \cup \Gamma_2} = \emptyset$.



Then u satisfies (32)-(34) and we apply Lemma 6.1 to u to conclude that $u = 0$ on $U_- = U \cap \{x_3 < 0\}$. Thus $u = 0$ on an open subset of $\mathbb{R}^{3-} \setminus \overline{\Gamma_1 \cup \Gamma_2}$. But u is analytic on $\mathbb{R}^{3-} \setminus \overline{\Gamma_1 \cup \Gamma_2}$ since

$$u_i = \int_{\Gamma_i} \left[\frac{\partial \Phi}{\partial \hat{n}(y)}(x, y) + \frac{\partial \Phi}{\partial \hat{n}(y)}(x, \overline{y}) \right] g_i dS(y), \ x \in \mathbb{R}^{3-} \setminus \overline{\Gamma},$$

so $u = 0$ everywhere in $\mathbb{R}^{3-} \setminus \overline{\Gamma_1 \cup \Gamma_2}$. Hence $u_1 = u_2$ on $\mathbb{R}^{3-} \setminus \overline{\Gamma_1 \cup \Gamma_2}$. Arguing by contradiction, suppose that $\overline{\Gamma}_1$ is not a subset of $\overline{\Gamma}_2$. Then $\exists y \in \overline{\Gamma}_1$ such that $y \notin \overline{\Gamma}_2$. Since $\overline{\Gamma}_2^C$ is open, $\exists r > 0$ such that $B(y, r) \subset \overline{\Gamma}_2^C$. Thus $B(y, r) \cap \overline{\Gamma}_2 = \emptyset$. Since $y \in \overline{\Gamma}_1$, $\exists y_0 \in B(y, r) \cap \overline{\Gamma}_2^C \cap \Gamma_1$, since $B(y, r) \cap \overline{\Gamma}_2^C$ is an open set containing y. But $\text{supp}(g_1) = \overline{\Gamma}_1$, so g_1 is not uniformly zero on Γ_1. Thus $\int_{B(y_0,r)\cap\Gamma_1} [u_1(y_0)] \neq 0$ but $\int_{B(y_0,r)\cap\Gamma_1} [u_2(y)] = 0$. This contradicts the fact that $u_1 = u_2$ in $\mathbb{R}^{3-} \setminus \overline{\Gamma_1 \cap \Gamma_2}$. To avoid contradiction, we conclude that $\overline{\Gamma}_1 \subset \overline{\Gamma}_2$. Switching the roles of $\overline{\Gamma}_1$ and $\overline{\Gamma}_2$, we see that $\overline{\Gamma}_1 \supset \overline{\Gamma}_2$. Since Γ_1 and Γ_2 are C^2 open surfaces $\overline{\Gamma_1} = \overline{\Gamma_2}$ whose boundaries are closed curves, implies that $\Gamma_1 = \Gamma_2$. We also know that $[u_1] = [u_2]$ across $\Gamma_1 = \Gamma_2$, so $g_1 = g_2$. ■

7 Stability Results

Recall the notation

$$G(x, y, \hat{n}) = \frac{\partial \Phi(x, y)}{\partial \hat{n}(y)} = \nabla_y \Phi(x, y) \cdot n.$$

In the rest of this thesis we extend this notation to the case of any unit vector v: we set,

$$G(x, y, v) = \nabla_y \Phi(x, y) \cdot v.$$

Lemma 7.1. *Let Γ be an open surface in $\mathbb{R}^3$ included in the plane $x_3 = 0$ and let $g \in C_c^\infty(\Gamma)$. Then g satisfies the following jump formulas across Γ.*

$$[\int_\Gamma G(x, y, e_3)g(y)dy_1dy_2] = g(x), \tag{35}$$

$$[\int_\Gamma (\partial_{y_1} G)(x, y, e_3)g(y)dy_1dy_2] = -\partial_{x_1} g(x), \tag{36}$$

$$[\int_\Gamma G(x, y, e_1)g(y)dy_1dy_2] = 0, \tag{37}$$

$$[\int_\Gamma G(x, y, e_2)g(y)dy_1dy_2] = 0, \tag{38}$$

$$[\int_\Gamma (\partial_{y_3} G)(x, y, e_3)g(y)dy_1dy_2] = 0, \tag{39}$$

$$[\partial_{x_1} \int_\Gamma G(x, y, e_3)g(y)dy_1dy_2] = \partial_{x_1} g(x), \tag{40}$$

$$[\partial_{x_2} \int_\Gamma G(x, y, e_3)g(y)dy_1dy_2] = \partial_{x_2} g(x), \tag{41}$$

$$[\partial_{x_3} \int_\Gamma G(x, y, e_3)g(y)dy_1dy_2] = 0. \tag{42}$$

Proof. We can perform a change of variables by translation so that $x = 0$. By Taylor's Theorem, we know that

$$g(y_1, y_2) = g(0, 0) + g_{y_1}(0, 0)y_1 + g_{y_2}(0, 0)y_2 + O(y_1^2 + y_2^2). \tag{43}$$

Clearly

$$G(x, y, e_3) = \partial_{y_3}\Phi(x, y) = \frac{x_3 - y_3}{4\pi |x - y|^3}.$$

Fix $\varepsilon > 0$. Then we apply a change of variables $v = r^2 + x_3^2$ to find that

$$\begin{aligned}
&\lim_{x_3 \to 0^+} \int_{B(0,\varepsilon)} (G(x_1, x_2, x_3, y, e_3) - G(x_1, x_2, -x_3, y, e_3)g(0, 0)dy_1dy_2 \\
&= \lim_{x_3 \to 0^+} g(0, 0) \int_0^{2\pi} 1 d\theta \int_0^\varepsilon \frac{x_3 r}{2\pi(r^2 + x_3^2)^{3/2}} dr = g(0, 0) \\
&= \lim_{x_3 \to 0^+} g(0, 0) \int_{x_3^2}^{x_3^2+\varepsilon^2} \frac{x_3}{2v^{3/2}} dv = g(0, 0)
\end{aligned} \tag{44}$$

As $\int_0^{2\pi}\sin(\theta)d\theta = \int_0^{2\pi}\cos(\theta)d\theta = 0$, we know that

$$\begin{aligned}
&\lim_{x_3\to 0^+}\int_{B(0,\varepsilon)} (G(x_1,x_2,x_3,y,e_3) - G(x_1,x_2,-x_3,y,e_3)y_1 g_{y_1}(0,0)dy_1dy_2 \\
&\quad = \lim_{x_3\to 0^+} g_{y_1}(0,0)\int_0^{2\pi}\cos(\theta)d\theta\int_0^{\varepsilon}\frac{x_3r^2}{2\pi(r^2+x_3^2)^{3/2}}dr = 0,
\end{aligned} \tag{45}$$

$$\begin{aligned}
&\lim_{x_3\to 0^+}\int_{B(0,\varepsilon)} (G(x_1,x_2,x_3,y,e_3) - G(x_1,x_2,-x_3,y,e_3)y_2 g_{y_2}(0,0)dy_1dy_2 \\
&\quad = \lim_{x_3\to 0^+} g_{y_2}(0,0)\int_0^{2\pi}\sin(\theta)d\theta\int_0^{\varepsilon}\frac{x_3r^2}{2\pi(r^2+x_3^2)^{3/2}}dr = 0.
\end{aligned} \tag{46}$$

By (43)-(46), jump of the integral of $G(x,y,e_3)g(y)$ over any open ball in the $y_1 - y_2$ plane is $g(0,0)$. But the open surface $\Gamma \subset \mathbb{R}^2 \times \{0\}$ is a union of such balls, so have shown (35). Since $g \in C_c^\infty(\Gamma)$, we can apply integration parts for $x \notin \Gamma$ to find that

$$\int_\Gamma (\partial_{y_1}G)(x,y,\hat{n})g(y)dy_1dy_2 = -\int_\Gamma G(x,y,\hat{n})(\partial_{y_1}g)(y)dy_1dy_2.$$

Then we can use (35) to derive (36). We compute

$$G(x,y,e_1) = \partial_{y_1}\Phi(x,y) = \frac{x_1-y_1}{4\pi\left|x-y\right|^{3/2}}.$$

Since this is even with respect to x_3 when $y_3 = 0$, so (37) follows. Equation (38) holds by the same argument. We know that

$$(\partial_{y_3}G)(x,y,e_3) = \partial_{y_3}^2\Phi(x,y) = \frac{3(x_3-y_3)^2}{4\pi\left|x-y\right|^5} \tag{47}$$

is even with respect to x_3 for in the $y_1 - y_2$ plane. This implies (39). We note that

$$\partial_{x_1}G(x_1,x_2,x_3,y,e_3) = \frac{3(y_3-x_3)(y_1-x_1)}{4\pi\left|x-y\right|^5}$$

As $\int_0^{2\pi}\cos(\theta)\sin(\theta)d\theta = \int_0^{2\pi}\cos(\theta)d\theta = 0$, we know that

$$\begin{aligned}
&\lim_{x_3\to 0^+}\partial_{x_1}\int_{B(0,\varepsilon)}\left(G(x_1,x_2,x_3,y,e_3) - G(x_1,x_2,-x_3,y,e_3)\right)g(0,0) \\
&\quad = g(0,0)\int_0^{2\pi}\cos(\theta)d\theta\int_0^{\varepsilon}\frac{6x_3r^2}{4\pi\left(r^2+x_3^2\right)^{5/2}}dr = 0,
\end{aligned}$$

$$\lim_{x_3\to 0^+} \partial_{x_1} \int_{B(0,\varepsilon)} \left(G(x_1,x_2,x_3,y,e_3) - G(x_1,x_2,-x_3,y,e_3)\right) y_2 g_{y_2}(0,0)$$
$$= \int_0^{2\pi} \cos(\theta)\sin(\theta)d\theta \int_0^{\varepsilon} \frac{6x_3 r^3}{4\pi(r^2+x_3^2)^{5/2}} dr = 0.$$

We can apply a change of variables $v = r^2 + x_3^2$ to get

$$\begin{aligned}
&\lim_{x_3\to 0^+} \partial_{x_1} \int_{B(0,\varepsilon)} \left(G(x_1,x_2,x_3,y,e_3) - G(x_1,x_2,-x_3,y,e_3)\right) y_1 g_{y_1}(0,0)\\
&= \lim_{x_3\to 0^+} g_{y_1}(0,0) \int_0^{2\pi} \cos^2(\theta)d\theta \int_0^{\varepsilon} \frac{6x_3 r^3}{4\pi(r^2+x_3^2)^{5/2}} dr\\
&= \lim_{x_3\to 0^+} g_{y_1}(0,0)(\pi) \int_{x_3^2}^{x_3^2+\varepsilon^2} \left(\frac{3x_3(v-x_3^2)}{4\pi v^{5/2}}\right) dv\\
&= \lim_{x_3\to 0^+} g_{y_1}(0,0) \left(\int_{x_3^2}^{x_3^2+\varepsilon^2} \left(\frac{3x_3}{4v^{3/2}} - \frac{3x_3^3}{4v^{5/2}}\right) dv\right)\\
&= \lim_{x_3\to 0^+} g_{y_1}(0,0) \left(-\frac{3x_3}{2\sqrt{v}} + \frac{x_3^3}{2\sqrt{v}^3}\Bigg|_{x_3^2}^{x_3^2+\varepsilon^2}\right)\\
&= g_{y_1}(0,0).
\end{aligned}$$

We repeat the argument used to derive (35) to conclude (40). Formula (41) follows by symmetry. We know that for $y_3 = 0$

$$\partial_{x_3} G(x,y,e_3) = \frac{-3x_3^2}{4\pi\left|x-y\right|^5}$$

is even with respect to x_3. This justifies (42). ■

Lemma 7.2. *Let* Γ *be as in lemma 7.1 and* $g \in H_0^1(\Gamma)$ *. Then the jump formulas* (35)*,* (37)*, and* (38) *still hold in the* $H_0^1(\Gamma)$ *norm, while the jump formulas* (36) *and* (39) *hold as continuous linear operations from* $H_0^1(\Gamma)$ *to* $L^2(\Gamma)$*. The same jump formulas hold for* H *in place of* G*.*

Proof. The result follows from the density of $H_0^1(\Gamma)$ in $C_c^\infty(\Gamma)$ and the smoothness of $(H - G)(x,y,\hat{n})$ for all $x,y \in \mathbb{R}^{3-}$. ■

Lemma 7.3. *Let* Γ *be a planar open surface in* $\mathbb{R}^3$ *(this time not necessarily included in the plane* $x_3 = 0$*) and let be* $g \in H_0^1(\Gamma)$*. Let* $\hat{t}$ *be a fixed unit vector parallel to* Γ*. Then* g

satisfies the following jump formulas across Γ*,*

$$[\int_{\Gamma} G(x,y,\hat{n})g(y)dS(y)] = g(x), \tag{48}$$

$$[\int_{\Gamma} (\frac{\partial_y G}{\partial \hat{t}})(x,y,\hat{n})g(y)dS(y)] = -\partial_{\hat{t}} g(x), \tag{49}$$

$$[\int_{\Gamma} G(x,y,\hat{t})g(y)dS(y)] = 0, \tag{50}$$

$$[\int_{\Gamma} (\frac{\partial_y G}{\partial \hat{n}})(x,y,\hat{n})g(y)dS(y)] = 0, \tag{51}$$

$$[\partial_{\hat{t}} \int_{\Gamma} G(x,y,\hat{n})g(y)dS(y)] = \partial_{\hat{t}} g(x), \tag{52}$$

$$[\partial_{\hat{n}} \int_{\Gamma} G(x,y,\hat{n})g(y)dS(y)] = 0. \tag{53}$$

Proof. Since the fundamental solution Φ given by definition 3.3 satisfies $\Phi(Tx,Ty) = \Phi(x,y)$, for all x and y in $\mathbb{R}^3$ such that $x \neq y$ where T is any rotation or translation of $\mathbb{R}^3$, this result is a straightforward generalization of lemma 7.2. ■

8 Lipschitz stability theorem for a fixed slip

For a closed rectangle R in the $x_3 = 0$ plane, we define the set $\Gamma_{a,b,d} = \{(x_1, x_2, ax_1 + bx_2 + d)|(x_1, x_2) \in R\}$. For any triplet $m = (a,b,d) \in \mathbb{R}^3$ we set $\Gamma_m = \Gamma_{a,b,d}$. Let B be a set of triplets (a,b,d) such that $\Gamma_{a,b,d} \subset \mathbb{R}^{3-}$. We assume that B is a closed and bounded subset of $\mathbb{R}^3$ so that

$$\inf_{m \in B} d(\Gamma_m, \{x_3 = 0\}) > 0. \tag{54}$$

We set $\hat{n} = \frac{(-a,-b,1)}{\sqrt{a^2+b^2+1}}$ and $\sigma_m = \sqrt{a^2+b^2+1}$ to be the normal vector and surface element of Γ_m. Let $H_0^1(R)$ be the space of functions g on R with Sobolev H_0^1 regularity. Let V be a relatively open subset of $\{x_3 = 0\}$. We define the operator $A_m : H_0^1(R) \to L^2(V)$ by

$$(A_m(g))(x) = \int_R H(x_1, x_2, x_3, y_1, y_2, ay_1 + by_2 + d, \hat{n})g(y_1, y_2)\sigma_m dy_1 dy_2. \tag{55}$$

Clearly A_m is linear and continuous. It can also be shown that A_m is compact: this can be done by considering the set $\{A_m g : g \in H_0^1(R), \|g\| < 1\}$, which is a set of equicontinuous functions, and by an application of Ascoli's theorem.

We know from Theorem 6.1 that A_m is injective. We fix a non-zero $h \in H_0^1(R)$ and define a linear function $\phi : B \to L^2(V)$ by

$$\phi(m) = \left(x \to \int_R H(x, y_1, y_2, ay_1 + by_2 + d) h(y_1, y_2) \sigma_m dy_1 dy_2 \right). \tag{56}$$

Due to the regularity of the Green's function H, we know that ϕ in analytic in m. Theorem 6.1 implies that ϕ is injective. We will show that the inverse of ϕ defined on $\phi(B)$ and valued in B is of class C^1 by applying the inverse function Theorem. As a consequence, ϕ^{-1} is Lipschitz continuous, which will yield our main stability estimate.

Theorem 8.1. *Assume that the set of admissible geometries B is such that for any $d < 0$, $(0, 0, d) \notin B$, in other words no horizontal faults are allowed. Fix a non-zero $h \in H_0^1(R)$ and define the function ϕ from B to $L^2(V)$ by (56). Then there is a positive constant C such that*

$$C|m - m'| \leq \|\phi(m) - \phi(m')\|_{L^2(V)} \tag{57}$$

for all m and m' in B.

Remark: The constant C in estimate (57) depends on the fixed slip h and on the compact set of values of the geometry parameters B, but is otherwise independent of m and m'.

Proof. Fix $m \in B$. Arguing by contradiction, suppose that there is an $m \in B$ such that $\nabla \phi(m)$ does not have full rank. Then there is a non-zero vector (c_1, c_2, c_3) such that

$$c_1 \frac{\partial}{\partial a} \phi(m) + c_2 \frac{\partial}{\partial b} \phi(m) + c_3 \frac{\partial}{\partial d} \phi(m) = 0. \tag{58}$$

We note that $\hat{n}\sigma$ simplifies to $(-a, -b, 1)$. Since

$$H(x, y, \hat{n}) = \frac{\partial \Phi(x_1, x_2, x_3, y)}{\partial \hat{n}(y)} - \frac{\partial \Phi(x_1, x_2, -x_3, y)}{\partial \hat{n}(y)}$$

is linear in $\hat{n}$, we can apply the chain rule with $y_3 = ay_1 + by_2 + d$ to find that

$$\begin{aligned} \frac{\partial}{\partial a} H(x, y, \hat{n}\sigma) &= \frac{\partial y_3}{\partial a} (\partial_{y_3} H)(x, y, \hat{n}\sigma) + H\left(x, y, \frac{\partial(\hat{n}\sigma)}{\partial a}\right) \\ &= y_1 (\partial_{y_3} H)(x, y, \hat{n}\sigma) - H(x, y, e_1). \end{aligned} \tag{59}$$

Similarly,

$$\frac{\partial}{\partial b} H(x, y, \hat{n}\sigma) = y_2(\partial_{y_3} H)(x, y, \hat{n}\sigma) - H(x, y, e_2) \tag{60}$$

and

$$\frac{\partial}{\partial d} H(x, y, \hat{n}\sigma) = (\partial_{y_3} H)(x, y, \hat{n}\sigma). \tag{61}$$

Define $f(y_1, y_2) = c_1 y_1 + c_2 y_2 + c_3$. Since h is independent of a, b, and d, we substitute (59)-(61) into (58) to find that

$$\begin{aligned} &\int_R (\partial_{y_3} H)(x, y_1, y_2, ay_1 + by_2 + d, \hat{n}) h(y_1, y_2) f(y_1, y_2) \sigma dy_1 dy_2 \\ &\qquad - \int_R H(x, y_1, y_2, ay_1 + by_2 + d, \nabla f) h(y_1, y_2) dy_1 dy_2 = 0 \end{aligned} \tag{62}$$

for all $x \in W$. Set $w(x)$ to be the left hand side of (62) where x has been extended to $\mathbb{R}^{3-} \setminus \Gamma_m$. We will now show that w is zero in $\mathbb{R}^{3-} \setminus \Gamma_m$. Since ∂_{y_3} and ∂_{x_i} commute, we know from the definition of the Green's function that w satisfies the Laplace Equation. We also know that w is 0 on V thanks to (58). By construction of the Green's tensor H, we know that for any x on the plane $x_3 = 0$, any $y \in \mathbb{R}^{3-}$, and any fixed vector $p \in \mathbb{R}^3$,

$$\partial_{x_3} H(x, y, p) = 0.$$

Thus we can take a ∂_{y_3} derivative and commute it with ∂_{x_3} to obtain

$$\partial_{x_3} \partial_{y_3} H(x, y, p) = 0.$$

It follows that $\partial_{x_3} w$ is also zero in W. We apply Lemma 6.1 to conclude that $w = 0$ everywhere in $\mathbb{R}^{3-} \setminus \Gamma_m$. In particular, the jump of w across Γ_m must be zero. As mentioned earlier, $H(x, y, p) - G(x, y, p)$ is smooth for any $x, y \in \mathbb{R}^{3-}$ and any fixed vector p in $\mathbb{R}^3$, therefore the jump across Γ_m of

$$\begin{aligned} &\int_R (\partial_{y_3} G)(x, y_1, y_2, ay_1 + by_2 + d, \hat{n}) h(y_1, y_2) f(y_1, y_2) \sigma dy_1 dy_2 \\ &\qquad - \int_R G(x, y_1, y_2, ay_1 + by_2 + d, \nabla f) h(y_1, y_2) dy_1 dy_2 \end{aligned} \tag{63}$$

is also zero.

Let α and β be in $\mathbb{R}$ and $\hat{t}$ a unit vector in $\mathbb{R}^3$ parallel to to Γ_m such that,

$$e_3 = \alpha\hat{n} + \beta\hat{t}.$$

Then (63) can be rewritten as

$$\begin{aligned}
\alpha \int_R (\partial_{\hat{n}} G)(x, y_1, y_2, ay_1 + by_2 + d, \hat{n}) h(y_1, y_2) f(y_1, y_2) \sigma dy_1 dy_2 \\
+ \beta \int_R (\partial_{\hat{t}} G)\,(x, y_1, y_2, ay_1 + by_2 + d, \hat{n})\, h(y_1, y_2) f(y_1, y_2) \sigma dy_1 dy_2 \\
- \int_R G(x, y_1, y_2, ay_1 + by_2 + d, \nabla f) h(y_1, y_2) dy_1 dy_2,
\end{aligned} \tag{64}$$

which again is zero for x in $\mathbb{R}^{3-} \setminus \overline{\Gamma_m}$. We now apply lemma 7.3: to be more precise, respectively formula (49) and (51) to the first and second terms of (64) and formulas (48) and (50) to the third term of (64) to obtain,

$$-\beta(\frac{\partial}{\partial\hat{t}} hf) - \frac{1}{\sigma}(\nabla f \cdot \hat{n}) h = 0 \tag{65}$$

Now we recall our assumption on B, the set of admissible geometries: it is such that for any $d < 0$, $(0, 0, d) \notin B$, in other words no horizontal faults are allowed. It follows that $\beta \neq 0$. As h is in $H_0^1(R)$, equation (65) implies that h is zero: this is due to lemma 3.3 in [9]. We conclude that we have contradicted the assumption $h \neq 0$ and thus $\nabla\phi(m)$ has full rank, for all m in the interior of B.

Now, since $\nabla\phi(m)$ has full rank, the inverse function theorem guarantees that ϕ defines a C^1 diffeomorphism from an open neighborhood U_m of m to its image by ϕ in $L^2(V)$. As the inverse of ϕ is also C^1 on $\phi(U_m)$, we find there is a ball $B(m, \epsilon_m)$ in $\mathbb{R}^3$and $C_m > 0$ such that for all m' in $B(m, \epsilon_m)$,

$$C_m\, |m - m'| \leq \|\phi(m) - \phi(m')\|_{L^2(V)} \tag{66}$$

Arguing by contradiction, assume that estimate (57) does not hold. Then there are two sequences p_n and q_n in B such that $p_n \neq q_n$ and

$$\lim_{n\to\infty} \frac{\|\phi(p_n) - \phi(q_n)\|_{L^2(V)}}{|p_n - q_n|} = 0. \tag{67}$$

As B is compact, without loss of generality we can assume that p_n converges to some $\tilde{m}$ in B and q_n converges to some $\tilde{\tilde{m}}$ in B. If $\tilde{m} = \tilde{\tilde{m}}$ this contradicts (66), while if $\tilde{m} \neq \tilde{\tilde{m}}$, this contradicts the uniqueness theorem 6.1. ■

9 Conclusion and perspectives for future work

We have shown in this thesis an existence and uniqueness theorem for our model PDE system in half space minus a fault, a uniqueness theorem for the related inverse problem, and a theorem regarding the stability of this inverse problem in case of planar faults. These three theorems will be the subject of a forthcoming publication. We envision to study in future work an extension of theorem 8.1 in the case of unknown slips, just as theorem 3.1 led to theorem in 4.1 in [9]. Eventually, it would be very interesting to work on the computational aspects of solving this plane or anti plane fault inverse problem. In the case of the full vector linear elasticity problem, this was done in [12] and in [10], where special random walks techniques were designed and implemented. The advantage of performing such a study in the plane or anti plane case would be that the related Green's function is orders of magnitude simpler to compute, thus it becomes easier to clearly focus on the development of related probability distribution functions for priors and posteriors, (in particular, with an efficient way of dealing with regularization constants, which we do not want to set equal to some arbitrary value), and associated random walks techniques, without being encumbered by the heavy computational cost of evaluating the Green function for that problem.

Appendices

A A proof of the jump formulas for double layer potential and continuous densities

In this thesis, we need to understand and use the jump formula for double layer potentials with densities with Sobolev regularity $H^{\frac{1}{2}}$. This jump formula was stated in lemma 3.5. As C^1 is dense in $H^{\frac{1}{2}}$, we first prove this formula for densities of class C^1. In fact, with no additional effort, this formula can be proved for continuous densities, which we set forth to do. A proof of that result can be found in [5], chapter 6. We present here a simpler proof, where for pedagogical reasons we provide ample details.

More background work is needed to cover the $H^{\frac{1}{2}}$ case: this can be done by studying the Hilbert transform, using Fourier transforms, and local charts to flatten the boundary of the surface. The $H^{\frac{1}{2}}$ case is beyond the scope of this thesis.

Lemma A.1.
$$\lim_{r\to 0^+}\int_{B(z,r)}\frac{1}{|x-y|^{d-1}}dy=0$$
uniformly in x and z in $\mathbb{R}^d$.

Proof. (Sketch) This is easily seen by applying the changes of variables $w=y-z$. ∎

Lemma A.2. *Let $\Phi(x,y)$ be the fundamental solution to the Laplace equation and D, $\hat{n}$, and ∂D be defined as in Lemma 3.1. Then*
$$\lim_{r\to 0^+}\int_{\partial D\cap B(z,r)}\left|\frac{\partial\Phi(x,y)}{\partial\hat{n}(y)}\right|=0$$
uniformly for all x and z in ∂D.

Proof. From Lemma 3.1, we know that
$$\left|\frac{\partial\Phi(x,y)}{\partial\hat{n}(y)}\right|\leq\frac{L\,|x-y|^2}{|x-y|^d},\ \forall x,y\in\partial D.$$
Thus we can apply local charts and the previous lemma to find that
$$\lim_{r\to 0^+}\int_{\partial D\cap B(z,r)}\left|\frac{\partial\Phi(x,y)}{\partial\hat{n}(y)}\right|=0$$
uniformly in x and z in ∂D. ∎

Lemma A.3. *Let Φ, D, ∂D, and $\hat{n}$ be defined as in Lemma A.2. Then*
$$\int_{\partial D}\frac{\partial\Phi(x,y)}{\partial\hat{n}(y)}dS(y)=\begin{cases}-1 & x\in D\\ -\frac{1}{2} & x\in\partial D\\ 0 & x\in\mathbb{R}^d\setminus\overline{D}\end{cases}.$$

Proof. For $x\in D$, we apply Green's Representation Theorem from Section 2.3 of [linear pde 2] to $u=-1$ to find that
$$-1=\int_{\partial D}\left\{\frac{\partial(-1)}{\partial\hat{n}}(y)\Phi(x,y)-(-1)\frac{\partial\Phi(x,y)}{\partial\hat{n}(y)}\right\}dS(y)=\int_{\partial D}\frac{\partial\Phi(x,y)}{\partial\hat{n}(y)}dS(y).$$
If $x\in\mathbb{R}^d\setminus\overline{D}$, $y\to\frac{\partial\Phi(x,y)}{\partial\hat{n}(y)}\in C^2(\overline{D})$ and we can use the Divergence Theorem to conclude

that

$$\int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \hat{n}(y)} dS(y) = \int_D \Delta_y \Phi(x,y) dy = 0. \tag{68}$$

Let $x \in \partial D$ and fix a small $r > 0$. We can define a closed surface S which is piecewise C^2 and globally Lipschitz by setting

$$S = \left(\partial D \cap (B(x,r))^C\right) \cup (\partial B(x,r) \cap D)$$

(see Figure 1). We also know that $x \in \mathbb{R}^d \setminus S$ so (68) with D replaced by S implies that

$$\int_S \frac{\partial \Phi(x,y)}{\partial \hat{n}(y)} dS(y) = 0.$$

Since $\partial D \cap B(x,r)^C$ and $\partial B(x,r) \cap D$ are disjoint we have

$$\int_{\partial D \cap B(x,r)^C} \frac{\partial \Phi(x,y)}{\partial \hat{n}(y)} dS(y) + \int_{\partial B(x,r) \cap D} \frac{\partial \Phi(x,y)}{\partial \hat{n}(y)} dS(y) = 0. \tag{69}$$

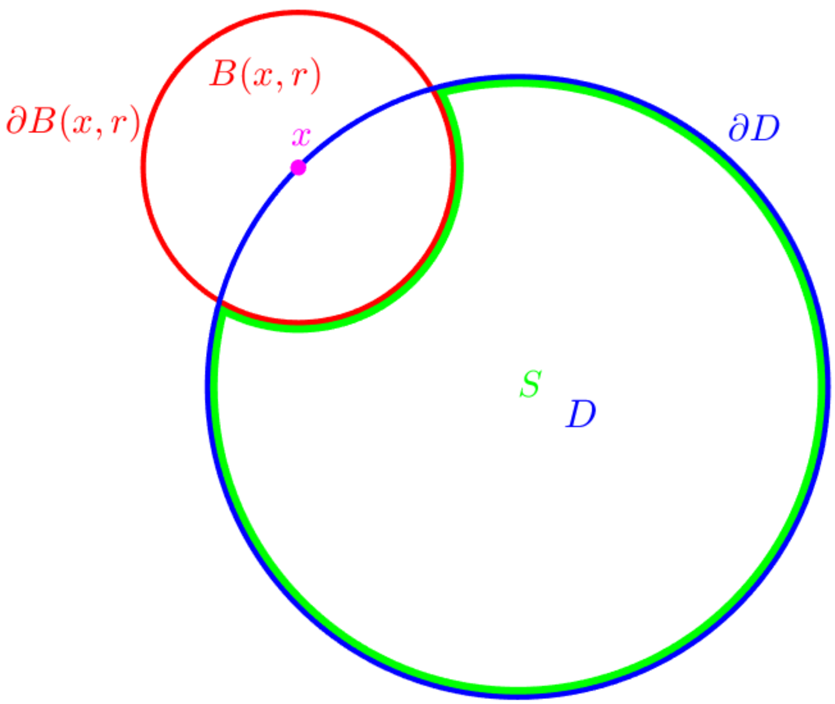


Figure 3: The set S defined as $\left(\partial D \cap (B(x,r))^C\right) \cup (\partial B(x,r) \cap D)$ is shown in green

As the radius of $x - y$ of the hypersphere $\partial B(x,r)$ points in the opposite direction of the

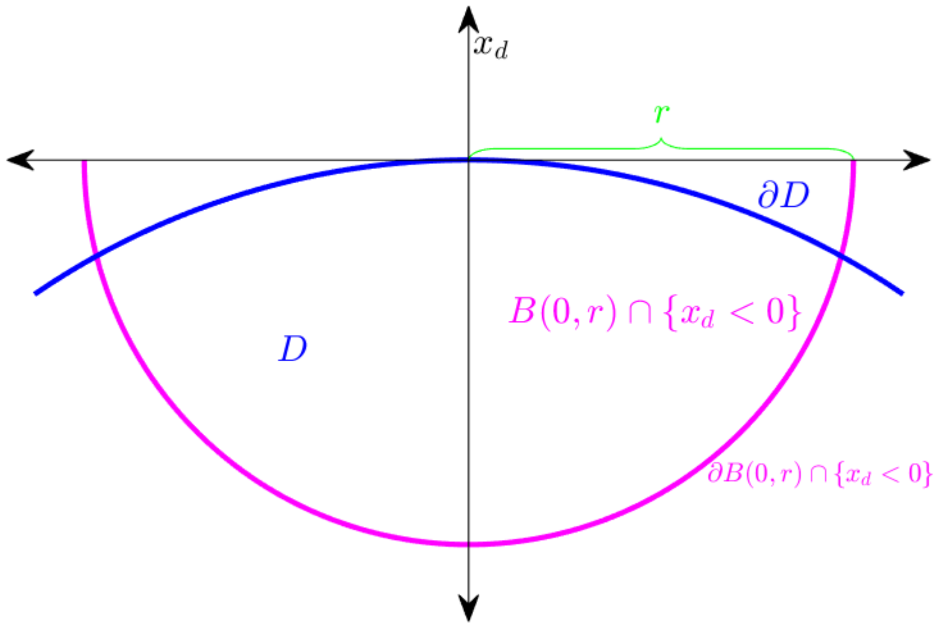


Figure 4: A local sketch of ∂D in the new coordinates

outer unit normal $\hat{n}(y)$, we have

$$\begin{aligned} \int_{\partial B(x,r)\cap D} \frac{\partial \Phi(x,y)}{\partial \hat{n}(y)} dS(y) &= -\int_{\partial B(x,r)\cap D} \frac{|x-y|\,|\hat{n}(y)|}{\omega_d\,|x-y|^d} dS(y) \\ &= -\frac{1}{\omega_d r^{d-1}}\,|\partial B(x,r)\cap D|\,. \end{aligned} \quad (70)$$

We can perform a change of coordinates by translation and rotation so that in the new coordinates $x = 0$ and the outward unit normal at x is e_d. Then the equation in ∂D in $B(x,r)$ is given by $x_d = \phi(x_1, ..., x_{n-1})$ for some function $\phi \in C^2(B(0,1))$. Since $x = 0$, we know that $\phi(0) = 0$ and $\nabla\phi(0)$ is parallel to e_d.
We apply Taylor's Formula for h in the plane $x_d = 0$ to get

$$\phi(h) = \phi(0) + \nabla\phi(0)\cdot h + O(|h|^2) = O(|h|^2),$$

so the distance from $\partial B(x,r)\cap\{x_d<0\}\cap D^C$ to $x_d = 0$ is $O(r^2)$. Thus $\partial B(x,r)\cap\{x_d < 0\}\cap D^C$ is contained in the region $\left\{x\in\mathbb{R}^d \,\middle|\, |x| = r,\ -Cr^2 \le x_d \le 0\right\}$ for some $C > 0$. The area of this region is $O(r^d)$, so we find that

$$\left|\partial B(x,r)\cap\{x_d<0\}\setminus\left(\partial B(x,r)\cap\{x_d<0\}\cap D^C\right)\right| = o(r^{d-1}).$$

Thus (70) implies that

$$\lim_{r\to 0^+}\int_{\partial B(x,r)\cap D}\frac{\partial\Phi(x,y)}{\partial\hat{n}(y)}dS(y)=\lim_{r\to 0^+}\frac{1}{\omega_d r^{d-1}}\left|\partial B(x,r)\cap\{x_d<0\}\right|=\frac{1}{2}.$$

We also know that $\frac{\partial\Phi(x,y)}{\partial\hat{n}(y)}dS(y)\in L^1(\partial D)$ for r small enough by Lemma A.2, so

$$\lim_{r\to 0^+}\int_{\partial D\cap B(x,r)^C}\frac{\partial\Phi(x,y)}{\partial\hat{n}(y)}dS(y)=\int_{\partial D}\frac{\partial\Phi(x,y)}{\partial\hat{n}(y)}dS(y).$$

Thus (69) implies that

$$\int_{\partial D}\frac{\partial\Phi(x,y)}{\partial\hat{n}(y)}dS(y)=-\frac{1}{2}.$$

■

Lemma A.4. *Let Φ, D, ∂D, and $\hat{n}$ be defined as in Lemma A.3 and let $\psi\in C(\partial D)$. Then*

$$u(x)=\int_{\partial D}\frac{\partial\Phi(x,y)}{\partial\hat{n}(y)}\psi(y)dS(y)$$

is continuous on ∂D.

Proof. Fix $z\in\partial D$. We will show that u is continuous at z. Fix $\varepsilon>0$. From Lemma A.2 we know that there is an $r>0$ such that

$$\int_{\partial D\cap B(z,r)}\left|\frac{\partial\Phi(x,y)}{\partial\hat{n}(y)}\right|\|\psi\|_\infty\,dS(y)<\frac{\varepsilon}{3},\ \forall x\in\partial D. \tag{71}$$

Let $x\in\partial D\cap B(z,\frac{r}{2})$. Then

$$\begin{aligned}|u(x)-u(z)|&\le\int_{\partial D\cap B(z,r)}\left|\frac{\partial\Phi(x,y)}{\partial\hat{n}(y)}-\frac{\partial\Phi(z,y)}{\partial\hat{n}(y)}\right|\|\psi\|_\infty\,dS(y)\\&\quad+\int_{\partial D\cap B(z,r)^C}\left|\frac{\partial\Phi(x,y)}{\partial\hat{n}(y)}-\frac{\partial\Phi(z,y)}{\partial\hat{n}(y)}\right|\|\psi\|_\infty\,dS(y)\\&\le\int_{\partial D\cap B(z,r)}\left|\frac{\partial\Phi(x,y)}{\partial\hat{n}(y)}\right|\|\psi\|_\infty\,dS(y)+\int_{\partial D\cap B(z,r)}\left|\frac{\partial\Phi(z,y)}{\partial\hat{n}(y)}\right|\|\psi\|_\infty\,dS(y)\\&\quad+\int_{\partial D\cap B(z,r)^C}\left|\frac{\partial\Phi(x,y)}{\partial\hat{n}(y)}-\frac{\partial\Phi(z,y)}{\partial\hat{n}(y)}\right|\|\psi\|_\infty\,dS(y)\end{aligned}\tag{72}$$

For $y\in B(z,r)^C$ we have $|z-y|>r$ and $|x-y|>\frac{r}{2}$, so Lemma 3.1 implies that there are constants C_1 and C_2 such that

$$\left|\frac{\partial\Phi(x,y)}{\partial\hat{n}(y)}-\frac{\partial\Phi(z,y)}{\partial\hat{n}(y)}\right|\|\psi\|_\infty \leq \frac{C_1}{|x-y|^{d-1}}+\frac{C_1}{|y-z|^{d-1}} \\ \leq \frac{C_2}{r^{d-1}} \in L^1(\partial D). \tag{73}$$

We also know that $x \to \left|\frac{\partial\Phi(x,y)}{\partial\hat{n}(y)}-\frac{\partial\Phi(z,y)}{\partial\hat{n}(y)}\right|\|\psi\|_\infty$ is continuous for $x \in B(z,\frac{r}{2})$, $y \in B(z,r)^C$, so we can apply the Dominated Convergence Theorem to find that $x \to \int_{\partial D\cap B(z,r)^C}\left|\frac{\partial\Phi(x,y)}{\partial\hat{n}(y)}-\frac{\partial\Phi(z,y)}{\partial\hat{n}(y)}\right|\|\psi\|_\infty\, dS(y)$ is continuous on $B(z,\frac{r}{2})$. Thus for x sufficiently close z we have

$$\int_{\partial D\cap B(z,r)^C}\left|\frac{\partial\Phi(x,y)}{\partial\hat{n}(y)}-\frac{\partial\Phi(z,y)}{\partial\hat{n}(y)}\right|\|\psi\|_\infty\, dS(y) < \frac{\varepsilon}{3} \tag{74}$$

Substituting (71) and (74) into (72), we conclude that $|u(x)-u(z)| < \varepsilon$ for x sufficiently close to z. We conclude that u is continuous on ∂D as required. ■

Lemma A.5. *Let Φ, D, ∂D, and $\hat{n}$ be defined as in Lemma A.3 and let $\psi \in C(\partial D)$. Define*

$$u(x) = \int_{\partial D}\frac{\partial\Phi(x,y)}{\partial\hat{n}(y)}\psi(y)dS(y),\ x \in D.$$

Then for any $z \in \partial D$ we have

$$\lim_{h\to 0^+} u(z-h\hat{n}(z)) = -\frac{1}{2}\psi(z)+u(z).$$

Proof. Let $z \in \partial D$ and fix $\varepsilon > 0$. Set $x = z - h\hat{n}(z)$. Then we know from Lemma A.3 that

$$\frac{1}{2}\psi(z) = \psi(z)+\psi(z)\int_{\partial D}\frac{\partial\Phi(z,y)}{\partial\hat{n}(y)}dS(y),$$

so

$$\begin{aligned}
&\left|u(z-h\hat{n}(z))-\left(u(z)-\frac{1}{2}\psi(z)\right)\right| \\
&\quad=\left|u(z-h\hat{n}(z))-u(z)+\psi(z)+\psi(z)\int_{\partial D}\frac{\partial\Phi(z,y)}{\partial\hat{n}(y)}dS(y)\right| \\
&\quad=\left|\int_{\partial D}\frac{\partial\Phi(x,y)}{\partial\hat{n}(y)}\psi(y)dS(y)-\int_{\partial D}\frac{\partial\Phi(z,y)}{\partial\hat{n}(y)}\psi(y)dS(y)+\psi(z)+\psi(z)\int_{\partial D}\frac{\partial\Phi(z,y)}{\partial\hat{n}(y)}dS(y)\right| \\
&\quad\leq\int_{\partial D}\left|\frac{\partial\Phi(x,y)}{\partial\hat{n}(y)}-\frac{\partial\Phi(z,y)}{\partial\hat{n}(y)}\right||\psi(y)-\psi(z)|\,dS(y)+\left|\psi(z)+\psi(z)\int_{\partial D}\frac{\partial\Phi(x,y)}{\partial\hat{n}(y)}dS(y)\right|.
\end{aligned}$$

For h small enough, $x \in D$, so Lemma A.3 implies that

$$\int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \hat{n}(y)} dS(y) = -1$$

and

$$\left| u(z - h\hat{n}(z)) - \left(u(z) - \frac{1}{2}\psi(z) \right) \right| \leq \int_{\partial D} \left| \frac{\partial \Phi(x,y)}{\partial \hat{n}(y)} - \frac{\partial \Phi(z,y)}{\partial \hat{n}(y)} \right| |\psi(y) - \psi(z)| \, dS(y).$$

Fix $r > 0$. For h small enough, we have

$$\begin{aligned}
\left| u(z - h\hat{n}(z)) - \left(u(z) - \frac{1}{2}\psi(z) \right) \right| &\leq \int_{\partial D \cap B(z,r)} \left| \frac{\partial \Phi(x,y)}{\partial \hat{n}(y)} - \frac{\partial \Phi(z,y)}{\partial \hat{n}(y)} \right| |\psi(y) - \psi(z)| \, dS(y) \\
&\quad + \int_{\partial D \cap B(z,r)^C} \left| \frac{\partial \Phi(x,y)}{\partial \hat{n}(y)} - \frac{\partial \Phi(z,y)}{\partial \hat{n}(y)} \right| |\psi(y) - \psi(z)| \, dS(y) \\
&\leq \max_{|y-z| \leq r} |\psi(y) - \psi(z)| \int_{\partial D \cap B(z,r)} \left(\left| \frac{\partial \Phi(x,y)}{\partial \hat{n}(y)} \right| + \left| \frac{\partial \Phi(z,y)}{\partial \hat{n}(y)} \right| \right) dS(y) \\
&\quad + 2 \|\psi\|_\infty \int_{\partial D \cap B(z,r)^C} \left| \frac{\partial \Phi(x,y)}{\partial \hat{n}(y)} - \frac{\partial \Phi(z,y)}{\partial \hat{n}(y)} \right| dS(y).
\end{aligned} \tag{75}$$

For h small enough, $z - h\hat{n}(z) \in D$ and $\frac{\partial \Phi(z-h\hat{n}(z),y)}{\partial \hat{n}(y)}$ is continuous in h for y in $\partial D \cap B(z,r)^C$. Thus, by taking h small enough we have

$$2 \|\psi\|_\infty \int_{\partial D \cap B(z,r)^C} \left| \frac{\partial \Phi(z - h\hat{n}(z),y)}{\partial \hat{n}(y)} - \frac{\partial \Phi(z,y)}{\partial \hat{n}(y)} \right| dS(y) < \frac{\varepsilon}{2},$$

i.e.,

$$2 \|\psi\|_\infty \int_{\partial D \cap B(z,r)^C} \left| \frac{\partial \Phi(x,y)}{\partial \hat{n}(y)} - \frac{\partial \Phi(z,y)}{\partial \hat{n}(y)} \right| dS(y) < \frac{\varepsilon}{2}. \tag{76}$$

We will next show that there is an $h_0 > 0$ such that

$$\int_{\partial D} \left| \frac{\partial \Phi(x,y)}{\partial \hat{n}(y)} \right| dS(y)$$

is uniformly bounded for $h \in [0, h_0]$ if $x = z - h\hat{n}(z)$. By Lemma 3.1 we have

$$\begin{aligned}
|x-y|^2 - \frac{1}{2}\left(|z-y|^2 + |x-z|^2 \right) &= \frac{1}{2}|z-y|^2 + \frac{1}{2}h^2 - 2(h\hat{n}(z) \cdot (y-z)) \\
&\geq \frac{1}{2}|z-y|^2 + \frac{1}{2}h^2 - 2hL|z-y|^2 .
\end{aligned}$$

Thus $|x-y|^2 \geq \frac{1}{2}\left(|z-y|^2+|x-z|^2\right)$ for h small enough. So we can apply Lemma 3.1 again to find that if h is small enough there is constant C_1 such that

$$\begin{aligned}
\left|\frac{\partial\Phi(x,y)}{\partial\hat{n}(y)}\right| &= \left|\frac{\hat{n}(y)\cdot(z-y)}{\omega_d\,|x-y|^d}+\frac{\hat{n}(y)\cdot(x-z)}{\omega_d\,|x-y|^d}\right| \\
&\leq \frac{L\,|z-y|^2}{\omega_d\,|x-y|^d}+\frac{|x-z|}{\omega_d\,|x-y|^d} \\
&\leq \frac{L\,|z-y|^2}{\omega_d\left(\frac{1}{2}(|z-y|^2+|x-z|^2)\right)^{d/2}}+\frac{|x-z|}{\omega_d\left(\frac{1}{2}(|z-y|^2+|x-z|^2)\right)^{d/2}} \\
&\leq C_1\left(\frac{1}{|z-y|^{d-2}}+\frac{|x-z|}{(|z-y|^2+|x-z|^2)^{d/2}}\right)
\end{aligned}$$

For h small enough, we have

$$\begin{aligned}
\int_{\partial D\cap B(z,r)^C}\left|\frac{\partial\Phi(x,y)}{\partial\hat{n}(y)}\right|dS(y) &\leq \int_{\partial D}\left|\frac{\partial\Phi(x,y)}{\partial\hat{n}(y)}\right|dS(y) \\
&\leq \int_{\partial D}C_1\left(\frac{1}{|z-y|^{d-2}}+\frac{|x-z|}{(|z-y|^2+|x-z|^2)^{d/2}}\right)dS(y).
\end{aligned}$$

Using local charts to map ∂D to the unit ball in $\mathbb{R}^{d-1}$, we find that if $r \leq 1$ and h is small enough, then there are constants C_1' and C_2' such that

$$\begin{aligned}
\int_{\partial D\cap B(z,r)}\left|\frac{\partial\Phi(x,y)}{\partial\hat{n}(y)}\right|dS(y) &\leq C_1'\int_0^r\left(\frac{1}{\rho^{d-2}}+\frac{|x-z|}{(\rho^2+|x-z|^2)^{d/2}}\right)\rho^{d-2}d\rho \\
&\leq C_2'r+C_2'\int_0^\infty\frac{\left(\frac{\rho}{|x-z|}\right)^{d-2}}{\left(\left(\frac{\rho}{|x-z|}\right)^2+1\right)^{d/2}}\frac{1}{|x-z|}d\rho \\
&\leq C_2'+C_2'\int_0^\infty\frac{\lambda^{d-2}}{(\lambda^2+1)^{d/2}}d\lambda.
\end{aligned}$$

We also know from Lemma A.2 that there is a constant C_2 such that for r small enough

$$\int_{\partial D\cap B(z,r)}\left|\frac{\partial\Phi(z,y)}{\partial\hat{n}(y)}\right|dS(y) \leq C_2.$$

Thus there is a constant C_0 such that for h and r small enough we have

$$\int_{\partial D\cap B(z,r)}\left(\left|\frac{\partial\Phi(x,y)}{\partial\hat{n}(y)}\right|+\left|\frac{\partial\Phi(z,y)}{\partial\hat{n}(y)}\right|\right)dS(y) \leq C_0.$$

Since ψ is continuous on ∂D and ∂D is compact, ψ is uniformly continuous on ∂D and we can fix an $r > 0$ small enough that

$$\max_{|y-z|\le r} |\psi(y) - \psi(z)| < \frac{\varepsilon}{2C_0}$$

and for h small enough

$$\max_{|y-z|\le r} |\psi(y) - \psi(z)| \int_{\partial D \cap B(z,r)} \left(\left| \frac{\partial \Phi(x,y)}{\partial \hat{n}(y)} \right| + \left| \frac{\partial \Phi(z,y)}{\partial \hat{n}(y)} \right| \right) dS(y) < \frac{\varepsilon}{2} \tag{77}$$

We combine (75)-(77) to find that for h small enough

$$\left| u(z - h\hat{n}(z)) - \left(u(z) - \frac{1}{2}\psi(z) \right) \right| < \varepsilon.$$

We conclude that

$$\lim_{h\to 0^+} u(z - h\hat{n}(z)) = u(z) - \frac{1}{2}\psi(z).$$

■

Lemma A.6. *Let Φ, D, ∂D, $\hat{n}$, and u be defined as in Lemma A.3 and let $\psi \in C(\partial D)$. Then for any $z \in \partial D$ we have*

$$\lim_{h\to 0^+} u(z + h\hat{n}(z)) = u(z) + \frac{1}{2}\psi(z).$$

Proof. This result follows from an argument similar to what was used to proof of Lemma A.5.

■

Lemma A.7. *Let Φ, D, ∂D, and $\hat{n}$ be defined as in Lemma A.3 and let $\psi \in C(\partial D)$. Then*

$$u(x) = \begin{cases} \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \hat{n}(y)} \psi(y) dS(y) & \text{if } x \in D \\ \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \hat{n}(y)} \psi(y) dS(y) - \frac{1}{2}\psi(x) & \text{if } x \in \partial D. \end{cases}$$

is continuous on $\overline{D}$.

Proof. Fix $z \in \partial D$. We will show that u is continuous at z. Let $x \in D$ and h be the distance from x to ∂D. Clearly h is smaller than $|x - z|$, so for x sufficiently close to z, h is small enough that the projection w of x onto ∂D is defined (see Section 2.5.6 of [6]). By Lemma A.5, we note that by making x close enough to z so that h is sufficiently small, we have $|u(x) - u(w)| < \frac{\varepsilon}{2}$. Since $|w - z| \leq |x - z| + h \leq 2\,|x - z|$ and u is continuous on ∂D by Lemma A.4, for x sufficiently close to z we have

$$|u(x) - u(z)| \leq |u(x) - u(w)| + |u(w) - u(z)| < \varepsilon.$$

Now fix $z \in D$. We will show that u is continuous at z. Since $y \to \frac{\partial \Phi(x,y)}{\partial \hat{n}(y)} \psi(y)$ is continuous on ∂D for $x \in D$, we can apply the Dominated Convergence Theorem to find that u is continuous at z. We conclude that u is continuous on $\overline{D}$ as required. ■

Lemma A.8. *Let Φ, D, ∂D, and $\hat{n}$ be defined as in Lemma A.3 and let $\psi \in C(\partial D)$. Then*

$$u(x) = \begin{cases} \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \hat{n}(y)} \psi(y) dS(y) & \text{if } x \in D \\ \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \hat{n}(y)} \psi(y) dS(y) - \frac{1}{2}\psi(x) & \text{if } x \in \partial D \\ \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \hat{n}(y)} \psi(y) dS(y) + \frac{1}{2}\psi(x) & \text{if } x \in \mathbb{R}^d \setminus \overline{D}. \end{cases}$$

is continuous on $\mathbb{R}^d$.

Proof. The result follows from an argument similar to what was used to prove the Lemma A.7. ■

Lemma A.9. *Let Φ, D, ∂D, and $\hat{n}$ be defined as in Lemma A.3 and let $\psi \in C(\partial D)$. Then the jump of*

$$u(x) = \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \hat{n}(y)} \psi(y) dS(y)$$

across ∂D is $\psi(x)$.

Proof. We apply Lemma A.5 and Lemma A.6 to find that

$$\begin{aligned} [u](z) &= \lim_{h \to 0^+} u(z + h\hat{n}(z)) - u(z - h\hat{n}(z)) \\ &= \lim_{h \to 0^+} (u(z + h\hat{n}(z)) - u(z)) + \lim_{h \to 0^+} (u(z) - u(z - h\hat{n}(z))) \\ &= \frac{1}{2}\psi(z) + \frac{1}{2}\psi(z) = \psi(z). \end{aligned}$$

■

B Classical results from functional analysis and PDE theory needed in this thesis

We will use the following lemma to prove Lemma 4.4 where we need to show that a particular function that is bounded in $H^1(\Omega)$ has a subsequence which converges weakly in $H^1(\Omega)$. The following lemma states a well known result which holds in the more general setting of reflexive Banach spaces, see [13]. Interestingly, a much simpler proof can be given in the case of Hilbert spaces as shown below.

Lemma B.1. *Let H be a separable Hilbert Space. Then every bounded sequence x_n in H has a weakly convergent subsequence. There is a result in Section 3.5 of [6] that proves that every bounded sequence in a reflexive Banach space has a weakly convergent subsequence, but this is more general that what we need.*

Proof. Since H is separable, there is a sequence y_m which is dense in H. We claim that for any $m \in \mathbb{N}$ there is a subsequence $x_{n_{k,m}}$ of x_n such that $\langle y_j, x_{n_{k,m}} \rangle$ converges as $k \to \infty$, $\forall j \leq m$. We will prove this by induction. We know that x_n is bounded, so

$$\exists M \in \mathbb{R}, \ \|x_n\| \leq M. \tag{78}$$

Thus

$$|\langle y_1, x_n \rangle| \leq \|y_1\| \, \|x_n\| \leq \|y_1\| \, M.$$

Hence $\langle y_1, x_n \rangle \in \mathbb{R}$ is bounded and has a convergent subsequence $\langle y_1, x_{n_{k,1}} \rangle$. Suppose that there is a subsequence $x_{n_{k,m}}$ of x_n such that $\langle y_j, x_{n_{k,m}} \rangle$ converges as $k \to \infty$ for $j \leq m$. We know from (78) that

$$\left|\langle y_{m+1}, x_{n_{k,m}} \rangle\right| \leq \|y_{m+1}\| \left\|x_{n_{k,m}}\right\| \leq \|y_{m+1}\| \, M.$$

Hence $\langle y_{m+1}, x_{n_{k,m}} \rangle \in \mathbb{R}$ is bounded and has a subsequence $\langle y_{m+1}, x_{n_{k,m+1}} \rangle$ which converges as $k \to \infty$. Since $\langle y_j, x_{n_{k,m}} \rangle$ converges as $k \to \infty$, $\forall j \leq m$ and its subsequence $\langle y_j, x_{n_{k,m+1}} \rangle$ converges as $k \to \infty$ for $j = m+1$, we find that $\langle y_j, x_{n_{k,m+1}} \rangle$ converges as $k \to \infty$ for $j \leq m+1$. We conclude that for every $m \in \mathbb{N}$ the sequence $\langle y_j, x_n \rangle$ has a subsequence $\langle y_j, x_{n_{k,m}} \rangle$ which converges as $k \to \infty$, $\forall j \leq m$ by induction. Set $x_{n_k} = x_{n_{k,k}}$. Then $\langle y_j, x_{n_k} \rangle$ converges $\forall j \in \mathbb{N}$. Fix $\varepsilon > 0$ and let $y \in H$. Then

$$\exists m \in \mathbb{N}, \|y - y_m\| < \varepsilon \tag{79}$$

by the density of the sequence y_m. We also know that

$$\exists N \in \mathbb{N},\ \forall j, k > N,\ \left|\langle y_m, x_{n_k}\rangle - \langle y_m, x_{n_j}\rangle\right| < \varepsilon \tag{80}$$

since $\langle y_m, x_{n_k}\rangle$ converges. Thus (78) implies that

$$\begin{aligned}
\left|\langle y, x_{n_k}\rangle - \langle y, x_{n_j}\rangle\right| &\leq |\langle y, x_{n_k}\rangle - \langle y_m, x_{n_k}\rangle| + \left|\langle y_m, x_{n_k}\rangle - \langle y_m, x_{n_j}\rangle\right| + \left|\langle y_m, x_{n_j}\rangle - \langle y, x_{n_j}\rangle\right| \\
&\leq \|y - y_m\| \, \|x_{n_k}\| + \left|\langle y_m, x_{n_k}\rangle - \langle y_m, x_{n_j}\rangle\right| + \|y - y_m\| \, \left\|x_{n_j}\right\| \\
&< (\varepsilon)(M) + \varepsilon + (\varepsilon)(M) = (2M+1)\varepsilon.
\end{aligned}$$

As $\varepsilon > 0$ is arbitrary, we conclude that $\langle y, x_{n_k}\rangle \in \mathbb{R}$ is Cauchy, so it converges. Thus we can define a linear functional $l : H \to \mathbb{R}$ by $l(y) = \lim\limits_{n\to\infty} \langle y, x_{n_k}\rangle$. Since $|\langle y, x_{n_k}\rangle| \leq \|y\| \, \|x_{n_k}\| \leq \|y\| \, M$, we know that $|l(y)| \leq M \|y\|$ so $l(y)$ is bounded. By the Riesz Representation Theorem, $\exists x \in H$ such that $\langle y, x\rangle = l(y) = \lim\limits_{n\to\infty} \langle y, x_{n_k}\rangle$. We conclude that the weak limit of x_{n_k} is x. ■

If Ω be an open and connected subset of $\mathbb{R}^n$, and f is a C^1 function defined in Ω such that $\nabla f = 0$, it is well known from elementary calculus that f must be constant in Ω. The issue is how to generalize that statement for functions f in H^1. The conclusion in that case may of course only hold almost everywhere. This property is used to show Lemma 4.4 in this thesis.

Lemma B.2. *Let Ω be an open and connected subset of $\mathbb{R}^n$, $f \in H^1_{loc}(\Omega)$. If $\nabla f = 0$, then $f(x) = f(y)$ for almost all $(x, y) \in \Omega \times \Omega$. For any open set subset Ω_1 of Ω with finite measure we have $f(x) = \frac{1}{|\Omega_1|}\int_{\Omega_1} f(y)dy$ a.e. in Ω.*

Proof. Let $u \in C^1(\overline{\Omega})$. Set $\Omega' = \{x \in \Omega | d(x, \Omega^C) > h\}$ for $h > 0$ small enough that Ω' is non-empty and connected. By the Fundamental Theorem of Calculus and the Cauchy Schwartz Inequality,

$$|u(x + he_i) - u(x)|^2 \leq h^2 \int_0^1 \left|\frac{\partial u}{\partial x_i}(x + the_i)\right|^2 dt.$$

Thus, by Fubini's Theorem,

$$\begin{aligned}\int_{\Omega'} |u(x+he_i)-u(x)|^2\,dx &\le h^2 \int_{\Omega'}\int_0^1 \left|\frac{\partial u}{\partial x_i}(x+the_i)\right|^2 dtdx \\ &= h^2 \int_0^1\int_{\Omega'} \left|\frac{\partial u}{\partial x_i}(x+the_i)\right|^2 dxdt \\ &= h^2 \left\|\frac{\partial u}{\partial x_i}\right\|^2_{L^2(\Omega')}.\end{aligned} \tag{81}$$

Let $(y_1, ..., y_n) \in B(0,h)$ and $x \in \Omega'$. Then $x+y \in \Omega$, so (81) implies that

$$\begin{aligned}|u(x+y)-u(x)|^2 &= \left|\sum_{k=1}^n u((x+y_1e_1+...+y_{k-1}e_{k-1})+y_ke_k) - u(x+y_1e_1+...+y_{k-1}e_{k-1})\right|^2 \\ &\le \sum_{k=1}^n y_k^2 \|\nabla u\|^2_{L^2(\Omega')} \\ &= h^2 \|\nabla u\|^2_{L^2(\Omega')}\end{aligned}$$

As $C^1(\overline{\Omega})$ is dense in $H^1(\Omega)$, $\exists u_n \in C^1(\overline{\Omega})$ such that $\|f-u_n\|_{H^1(\Omega)} \to 0$. By the converse of the Dominated Convergence Theorem, a subsequence u_{n_k} of u_n converges to f a.e. Since $|u_{n_k}(x+y)-u_{n_k}(x)|^2 \le h^2 \|\nabla u_{n_k}\|^2_{L^2(\Omega')}$, $\forall x \in \mathbb{R}^n$, $\forall y \in B(0,h)$, and $\nabla u_{n_k} \to 0$ in $L^2(\Omega)$, we have $|f(x+y)-f(y)| = 0$ for almost all $(x,y) \in \Omega' \times B(0,h)$. Thus $f(y) = f(x)$ for almost all $(x,y) \in \Omega' \times B(x,h)$ **(4)**. Set $M = u(x_0)$. Set $S = \{x \in \Omega' | u(x) = M\}$. Since S satisfies the open ball property by **(4)**, S is open. We also know that S is closed in Ω' since it is the inverse image of the compact set $\{M\}$ under the continuous function u. But Ω' is connected, so $S = \Omega'$ or $S = \emptyset$. And $x_0 \in S$, so $S = \Omega'$. Thus $u(x) = M$ on Ω'. Letting $h \to 0$, we find that $u(x) = M$ on Ω. Clearly $M = \frac{1}{|\Omega_1|}\int_{\Omega_1} M dy = \frac{1}{|\Omega_1|}\int_{\Omega_1} u(y)dy$. Thus $u(x) = \frac{1}{|\Omega_1|}\int_{\Omega_1} u(y)dy$, $\forall x \in \overline{\Omega}$. But $f = u$ a.e., so $f(x) = \frac{1}{|\Omega_1|}\int_{\Omega_1} u(y)dy = \frac{1}{|\Omega_1|}\int_{\Omega_1} f(y)dy$ for almost all $x \in \Omega$. ■

References

[1] Dascalu, C., Ionescu, I. R., & Campillo, M. (2000). Fault finiteness and initiation of dynamic shear instability. Earth and Planetary Science Letters, 177(3-4), 163-176.

[2] Folland, G. B. (1995). Introduction to partial differential equations. Princeton university press.

[3] Ionescu, I. R., & Volkov, D. (2008). Earth surface effects on active faults: An eigenvalue asymptotic analysis. Journal of computational and applied mathematics, 220(1-2), 143-162.

[4] Ionescu, I. R., & Volkov, D. (2006). An inverse problem for the recovery of active faults from surface observations. Inverse problems, 22(6), 2103.

[5] Kress, R., Mazya, V., & Kozlov, V. (1989). Linear integral equations (Vol. 82, p. 1). Berlin: Springer.

[6] Nédélec, J. C. (2001). Acoustic and electromagnetic equations: integral representations for harmonic problems. Springer Science & Business Media.

[7] Slaughter, W. S. (2012). The linearized theory of elasticity. Springer Science & Business Media.

[8] Tréves, F. (1975). Basic linear partial differential equations (Vol. 62). Academic press.

[9] Triki, F., & Volkov, D. (2019). Stability estimates for the fault inverse problem. Inverse Problems, 35(7), 075007.

[10] Volkov, D. (2019). Stochastic solutions to mixed linear and nonlinear inverse problems. arXiv preprint arXiv:1909.07200.

[11] Volkov, D., Voisin, C., & Ionescu, I. R. (2017). Reconstruction of faults in elastic half space from surface measurements. Inverse Problems, 33(5), 055018.

[12] Volkov, D., Sandiumenge, J. C. (2019). A stochastic approach to reconstruction of faults in elastic half space Inverse Problems & Imaging 13 (3), 479-511

[13] Yosida, K. (2012). Functional analysis. Springer Science & Business Media.