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MAJORANA TREATMENT OF SPIN-1 STATE

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Abstract

The objective of this project is to use the Majorana description of a spin-1 system to examine many of its properties. In the Majorana approach, an arbitrary pure state of a spin-1 system is represented by a pair of points on the unit sphere, or a pair of unit vectors. Spin-1 states can be of three types: those whose vectors are parallel, those whose vectors are antiparallel and those whose vectors make an arbitrary angle. An expression is derived for the squared overlap of two states in terms of their Majorana (or M-) vectors and used to work out all the orthogonalities and bases that are possible for the three geometrically distinct types of spin-1 states. Finally, a geometrical construction is given of a set of mutually unbiased bases for a spin-1 system. The applications of this result to problems in quantum information are mentioned.

Introduction:

The Majorana representation [1] of the states of an arbitrary spin system is not very familiar to physicists, yet it gives a geometrically useful picture of such states that has found many applications. Penrose [2] used the Majorana representation to investigate many of the properties of arbitrary spin systems and Zimba and Penrose [3] used the Majorana description of a spin-3/2 system to give a proof of Bell's non-locality theorem that does not involve the use of probabilities. Hanny [4] employed the Majorana representation to analyze the Berry phase for spin systems and Zimba [5] investigated the properties of anti-coherent spin-states using the Majorana representation.

In the Majorana approach, an arbitrary pure state of spin-$\frac{1}{2}n$ system is represented by a set of n unordered points on a unit sphere known as the Riemann sphere [2]. A state of a spin-1/2 particle is represented by a single point and that of a spin-1 particle by a pair of points or, equivalently, by a pair of unit vectors. Since a pair of vectors are involved, the states of a spin-1 system can be of three types: those whose vectors are parallel (which we will refer to as coherent states, or C-states), those whose vectors are anti-parallel (which we will refer to as anti-coherent states, or A-states) and those whose vectors make an arbitrary angle (which we will refer to as devious states, or D-states).

A fundamental property of any two spin states is the squared modulus of their inner product. A basic question one can ask about spin-1 states is this: what is the expression for the squared modulus of the inner product of the two states in terms of their Majorana vectors (or M-vectors)? An answer to this question seems to have been given only recently by Gould and Aravind [6]. This project provides a simple derivation of this expression and then uses it as the basis for a systematic investigation of all the kinematic properties of spin-1 states.

A basic question one can ask about spin-1 states is how their M-vectors should be related if they are to be orthogonal. This project gives a detailed answer to this question on the basis of the inner product formula mentioned above.

We will refer to a set of three mutually orthogonal states of a spin-1 system as a basis. One can ask how many geometrically distinct types of bases there are involving the C-,A- and D-states. This question is also answered in this project.

Finally we turn to the concept of mutually unbiased bases (or MUBs). Two different bases are said to be mutually unbiased if the squared overlap of any member of one basis with any member of the other is the same and equal to 1/d, where d is the dimension of the system; thus, for spin-1, the value of the squared overlap is 1/3. Mutually unbiased bases have been widely studied both for their foundational interest [7-9] and their applications to numerous quantum information protocols, particularly quantum key distribution [10-14]. It is known, for a spin-1 system, that the maximum number of mutually unbiased bases is 4. Algebraic constructions of MUBs have been given for a spin-1 system as well as for many higher systems [15-21]. This paper breaks new ground by giving a purely geometrical construction of MUBs in a spin-1 system on the basis of the inner product formula for spin-1 states. This construction is of interest in connection with a number of problems in quantum information, and this point will be discussed further in the concluding section of this report.

I. Spin-1/2 States

The Majorana approach represents a spin-1/2 state as a point on a unit sphere. In this chapter we give a quick review of the Majorana description of spin-1/2 states, as that will make it easier to understand the extension of the description to higher spin systems.

(a) Parameterization of States

Let $|z\rangle$ and $|-z\rangle$ be the normalized spin-up and spin-down states of a spin -1/2 particle along the z-axis. We can express the most general pure state of such a particle as

$$|\alpha\rangle = \frac{1}{\sqrt{1+|\alpha|^2}}\left[|z\rangle + \alpha|-z\rangle\right] \tag{1.1}$$

$$|a\rangle = \sqrt{\frac{1+a_z}{2}}\left[|z\rangle + \frac{a_x + ia_y}{1+a_z}|-z\rangle\right] \tag{1.2}$$

$$|\theta,\phi\rangle = \cos\frac{\theta}{2}|z\rangle + e^{i\phi}\sin\frac{\theta}{2}|-z\rangle \tag{1.3}$$

where the state is parameterized by the complex number α in (1.1) , by the unit vector $a = (a_x, a_y, a_z)$ in (1.2) or by the point with spherical coordinates (θ,ϕ) on the unit sphere in (1.3). These parameterizations are related to each other by the equations

$$a_x = \sin\theta\cos\phi, \quad a_y = \sin\theta\sin\phi \quad \text{and} \quad a_z = \cos\theta \tag{1.4}$$

$$\alpha = \tan\frac{\theta}{2}e^{i\phi} = \frac{a_x + ia_y}{1 + a_z} \qquad (1.5)$$

$$a_x = \frac{2\,\mathrm{Re}(\alpha)}{1+|\alpha|^2}, \quad a_y = \frac{2\,\mathrm{Im}(\alpha)}{1+|\alpha|^2}, \quad a_z = \frac{1-|\alpha|^2}{1+|\alpha|^2} \qquad (1.6)$$

The connection between α and a, expressed in (1.5), can be visualized by using a sphere, referred to as the Riemann sphere, which is shown in Figure 1. The Riemann sphere is a sphere of unit radius whose equatorial plane is a complex plane and whose center is the origin of that plane. We can project a point P (the tip of the unit vector a) onto the complex plane by drawing a straight line from the South Pole to P and seeing where it intersects the plane. The image P' of P in the plane is represented by the complex number α. This mapping between points on the sphere and points on the plane is called stereographic projection.
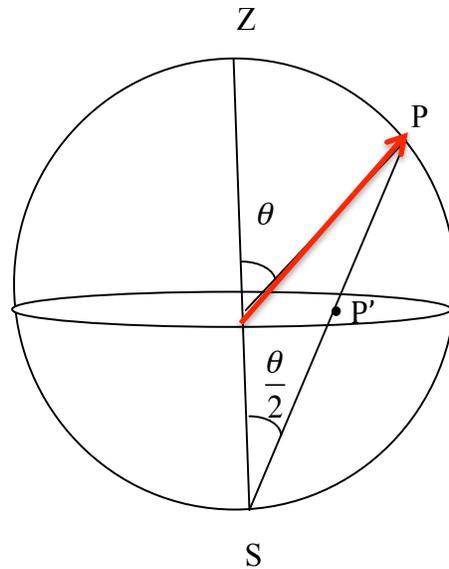
Figure 1: Stereographic Projection of the point P at (θ, φ) through the South Pole S on to the complex plane passing through its equator.

The kinematics and dynamics of pure spin-1/2 states can be studied either on the unit (Riemann or Bloch) sphere or in the complex plane. Equation (1.2) can be used to express the states on the unit sphere. On the other hand Equation (1.1) can be used to express the states as points in the complex plane. We will pass back and forth between these two representations and label the states as $|a\rangle$ or $|\alpha\rangle$, depending on the variable that is useful in a particular context.

(b) Overlap of two states:

Consider the normalized spin-1/2 states

$$|a\rangle = |\alpha\rangle = N_1 \left[|z\rangle + \alpha |-z\rangle \right], \quad \text{where} \quad N_1 = \frac{1}{\sqrt{1+|\alpha|^2}} = \sqrt{\frac{1+a_z}{2}} \tag{1.7}$$

and

$$|b\rangle = |\beta\rangle = N_2 \left[|z\rangle + \beta |-z\rangle \right], \quad \text{where} \quad N_2 = \frac{1}{\sqrt{1+|\beta|^2}} = \sqrt{\frac{1+b_z}{2}} \tag{1.8}$$

The overlap of these states can be expressed in the alternative forms

$$\langle b|a\rangle = \frac{1}{\sqrt{1+|\alpha|^2}\sqrt{1+|\beta|^2}} \left[1 + \beta^* \alpha \right] \tag{1.9}$$

$$= \frac{\sqrt{(1+a_z)(1+b_z)}}{2} \left[1 + b \bullet a \right] \tag{1.10}$$

$$= \cos\frac{\theta_b}{2} \cos\frac{\theta_a}{2} + e^{-i(\phi_b - \phi_a)} \sin\frac{\theta_b}{2} \sin\frac{\theta_a}{2} \tag{1.11}$$

,

where (1.9) and (1.11) express the overlap in terms of the complex parameters or the spherical coordinates of the vectors representing the states. (1.10) on the other hand expresses the overlap entirely in terms of the unit vectors of the states. The twisted product of two vectors, $b \bullet a$, is defined as

$$b \bullet a = \frac{b \,\mathrm{o}\, a + i(b \wedge a)}{(1 + b_z)(1 + a_z)} \tag{1.12}$$

with $b \,\mathrm{o}\, a = b_x a_x + b_y a_y$ and $b \wedge a = b_x a_y - b_y a_x$ are the "two-dimensional" dot and cross products of these vectors.

(c) Mutually Unbiased bases

A basis is a set of mutually orthogonal states equal in number to the dimension of the state space. For a spin-1/2 particle, any two orthogonal states form a basis. Two different bases, $|\psi_i\rangle$ and $|\phi_i\rangle$ with i=1,2 , are said to be mutually unbiased if

$\left|\langle \psi_i | \phi_j \rangle\right|^2 = 1/2$ for all $i, j = 1, 2$. A spin-1/2 system is well known to have a maximum of three mutually unbiased bases. One set of such bases is the spin-up and spin-down states along the z, x and y axes: $\{|z\rangle, |-z\rangle\}, \{|x\rangle, |-x\rangle\}$ and $\{|y\rangle, |-y\rangle\}$. For a spin-1/2 particle, oppositely directed vectors represent orthogonal states and the vectors pointing up and down along any three mutually orthogonal directions represent a set of mutually unbiased bases. The geometry of orthogonality, bases and mutually unbiased bases becomes much more complicated if one considers a spin-1 system, and it is one of the goals of this work to address this issue.

II. Spin-I States

In this section we will make use of spin-1/2 Majorana representation to parameterize a spin-1 state and derive an expression for the squared overlap of any two spin-1 states in terms of their M-vectors. We will make use of this expression to discuss orthogonality conditions, basis vectors and mutually unbiased bases for spin-1 states.

(a) Parameterization of states:

In the Majorana approach, a general (pure) state of a spin-1 particle is constructed as the symmetrized outer product of the states of a pair of fictitious spin-1/2 particles,

$$
|a_1, a_2\rangle = \frac{|a_1\rangle_1 \otimes |a_2\rangle_2 + |a_2\rangle_1 \otimes |a_1\rangle_2}{\sqrt{2\left(1 + \left|\langle a_2 | a_1 \rangle\right|^2\right)}} = \frac{|a_1\rangle_1 \otimes |a_2\rangle_2 + |a_2\rangle_1 \otimes |a_1\rangle_2}{\sqrt{3 + a_1 \cdot a_2}} \tag{2.1}
$$

where subscripts on the kets refer to the two spin-1/2 particles. The denominator in (2.1) ensures that the state is normalized, and the ket subscripts have been dropped from $\langle a_2 | a_1 \rangle$ because such products are the same for all spin-1/2 particles. Also $|a_1, a_2\rangle \equiv |a_2, a_1\rangle$ so the order of the state labels doesn't matter.

Three geometrically distinct types of spin-1 states are possible:

1. Those for which $a_1 = a_2$, or the state vectors are parallel. These states are referred as coherent states or C-States in our nomenclature.

2. Those for which $a_1 = -a_2$, or the states vectors are antiparallel. We will refer to these as anti-coherent states or A-States.

3. Those for which the vectors are neither parallel nor antiparallel but make an arbitrary angle with each other. We will refer to these as devious or D-states. Indeed both C- and A- states are limiting cases of D-states.

These three types of state are shown in Fig. 2. For each of these states we can define an axis. The axis for a C-state or A-state is the line along which all the vectors lie. The axis of a D-state axis is the line bisecting its vectors (i.e. the line making equal angles with them and in the same plane as them).

C-State   A-State   D-State
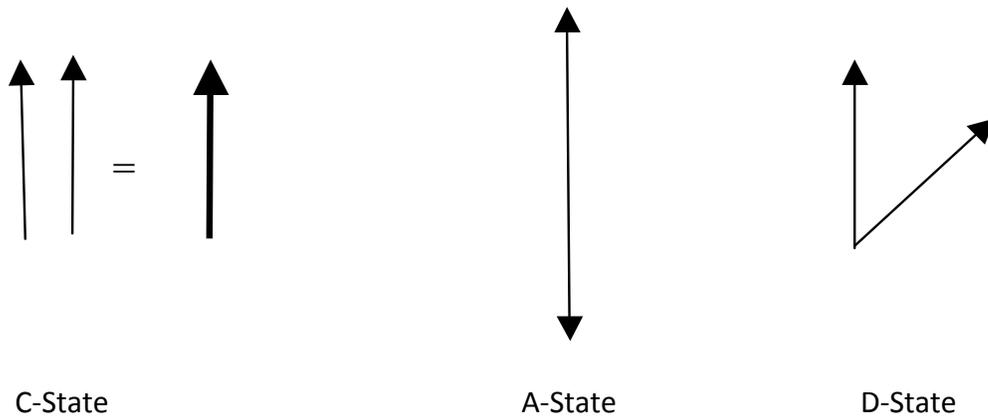
Figure 2: Spin-1 States

There exist SU(3) transformations that map any state of a spin-1 system into any other state. These transformations therefore map the above three types of states into each other. However the subgroup of SU(2) transformations generated by the angular momentum operators do not mix the different types of states, and it therefore makes sense to maintain the distinction between them.

From (2.1) one sees that the overlap of the C-state $\left|-a_1,-a_1\right\rangle \equiv \left|-a_1\right\rangle_1 \otimes \left|-a_1\right\rangle_2$ with the

D-state $\left|a_1,a_2\right\rangle$ vanishes:

$$\left\langle -a_1,-a_1 \middle| a_1,a_2 \right\rangle = 0 \tag{2.2}$$

as a result of the orthogonality property $\left\langle -a_1 \middle| a_1 \right\rangle = 0$ of a spin-1/2 state. Eq. (2.2)

implies that a D-state is orthogonal to a C-state whose vectors are anti-parallel to either

of its vectors. We will make use of this property at many points in our discussion.

One can recast (2.1) by substituting (1.1) and manipulating it as follows:

$$\left|a_1,a_2\right\rangle = \frac{\left[\left|z\right\rangle_1 + \alpha_1\left|-z\right\rangle_1\right] \otimes \left[\left|z\right\rangle_2 + \alpha_2\left|-z\right\rangle_2\right] + \left[\left|z\right\rangle_1 + \alpha_2\left|-z\right\rangle_1\right] \otimes \left[\left|z\right\rangle_2 + \alpha_1\left|-z\right\rangle_2\right]}{\sqrt{1+\left|\alpha_1\right|^2}\sqrt{1+\left|\alpha_2\right|^2}\sqrt{3+a_1\cdot a_2}}$$

$$= \frac{2\left|z\right\rangle_1\left|z\right\rangle_2 + \left(\alpha_1+\alpha_2\right)\left(\left|z\right\rangle_1\left|-z\right\rangle_2 + \left|-z\right\rangle_1\left|z\right\rangle_2\right) + 2\alpha_1\alpha_2\left|-z\right\rangle_1\left|-z\right\rangle_2}{\sqrt{1+\left|\alpha_1\right|^2}\sqrt{1+\left|\alpha_2\right|^2}\sqrt{3+a_1\cdot a_2}}$$

$$= \frac{\sqrt{\left(1+a_{1z}\right)\left(1+a_{2z}\right)}}{\sqrt{3+a_1\cdot a_2}}\left[\left|z,z\right\rangle + \frac{1}{\sqrt{2}}\left\{\frac{a_{1x}+ia_{1y}}{1+a_{1z}} + \frac{a_{2x}+ia_{2y}}{1+a_{2z}}\right\}\left|z,-z\right\rangle + \left(\frac{a_{1x}+ia_{1y}}{1+a_{1z}}\right)\left(\frac{a_{2x}+ia_{2y}}{1+a_{2z}}\right)\left|-z,-z\right\rangle\right]$$

$$\tag{3.2}$$

where we have introduced the following three states in the last line:

$$\left|z,z\right\rangle = \left|z\right\rangle_1\left|z\right\rangle_2 \tag{2.4}$$

$$\left|z,-z\right\rangle = \left|-z,z\right\rangle = \frac{1}{\sqrt{2}}\left(\left|z\right\rangle_1\left|-z\right\rangle_2 + \left|-z\right\rangle_1\left|z\right\rangle_2\right) \tag{2.5}$$

and $\quad\quad \left|-z,-z\right\rangle = \left|-z\right\rangle_1\left|-z\right\rangle_2 \tag{2.6}$

These states have spin component +1, 0 or -1 along the z-axis and are just the usual angular momentum basis state. We will sometimes refer to them as the z-basis.

In (2.1) we expressed a spin-1 state in terms of the vectors of its fictitious spin-1/2 components. In the last equation of (2.3), by contrast, we expand the state in the angular momentum basis. Both forms are useful, and we will use them both in our work. We will also express the state as either $\left|a_1, a_2\right\rangle$ or $\left|\alpha_1, \alpha_2\right\rangle$ depending on the parameterization we wish to emphasize. Sometimes one is given the components of a state in the angular momentum basis and one needs to work out its Majorana vectors; this can be done by solving a quadratic, as explained in Appendix A.

The overlap of two spin-1 states is of prime interest. Using the expression (2.3) and the twisted product definition from (1.12), we can express this overlap as

$$\left\langle b_1, b_2 \mid a_1, a_2 \right\rangle = \sqrt{\frac{\left(1+a_{1z}\right)\left(1+a_{2z}\right)\left(1+b_{1z}\right)\left(1+b_{2z}\right)}{\left(3+a_1 \cdot a_2\right)\left(3+b_1 \cdot b_2\right)}} \left[1+\frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2} a_i \bullet b_j +\left(a_1 \bullet b_1\right)\left(a_2 \bullet b_2\right)\right] \quad (2.7)$$

This expression involves only the real vectors of the states. However phase information is contained in it, as the twisted products are complex. The rotation matrix for spin-1 follows readily from (2.7) if one applies it to the angular states along two different axes related to each other by a sequence of three Euler angle rotations.

(b) Squared overlap of two spin-1 states

The overlap of the two spin-1 states $\left|a_1,a_2\right\rangle$ and $\left|b_1,b_2\right\rangle$ can be written, using (2.1), as

$$\left\langle b_1,b_2\middle|a_1,a_2\right\rangle = \frac{2}{\sqrt{\left(3+a_1\cdot a_2\right)\left(3+b_1\cdot b_2\right)}}\left[\left\langle b_1\middle|a_1\right\rangle\left\langle b_2\middle|a_2\right\rangle + \left\langle b_1\middle|a_2\right\rangle\left\langle b_2\middle|a_1\right\rangle\right],\qquad (2.8)$$

which looks a little different from (2.7) but is equivalent to it. We want to find an expression for the squared modulus of this overlap, denoted by P, in terms of the vectors characterizing the two states. The most efficient and easy way of doing this is to note that P must be a function of all the scalar products that can be constructed out of the vectors a_1, a_2, b_1 and b_2. It is evident from (2.8) that the form of this relationship is

$$P \equiv \left|\left\langle b_1,b_2\middle|a_1,a_2\right\rangle\right|^2 = \frac{n_1 + n_2L_1 + n_3L_2 + n_4Q_1 + n_5Q_2}{\left(3+a_1\cdot a_2\right)\left(3+b_1\cdot b_2\right)},\qquad (2.9)$$

where

$$L_1 = a_1\cdot a_2 + b_1\cdot b_2 \;,\quad L_2 = a_1\cdot b_1 + a_1\cdot b_2 + a_2\cdot b_1 + a_2\cdot b_2 \qquad (2.10)$$

$$Q_1 = \left(a_1\cdot b_1\right)\left(a_2\cdot b_2\right) + \left(a_1\cdot b_2\right)\left(a_2\cdot b_1\right)\;,\quad Q_2 = \left(a_1\cdot a_2\right)\left(b_1\cdot b_2\right) \qquad (2.11)$$

and $n_i\left(i=1,\cdots,5\right)$ are constants that remain to be determined. The numerator of (2.9) contains linear and quadratic terms in the scalar product, with the linear terms denoted by $L_{1,2}$ and the quadratic ones by $Q_{1,2}$. All these terms are invariant under an exchange of a_1 and a_2, or b_1 and b_2, or under an exchange of labels a and b. Using the known values of P in two simple limiting cases we can evaluate the constants $n_1,\cdots n_5$:

I. If $a_1 = b_1$ and $a_2 = b_2$, then P=1. Substituting these in (2.9) gives the equation

$$P = \frac{n_1 + 2n_3 + n_4 + 2\left(n_2 + n_3\right)x + \left(n_4 + n_5\right)x^2}{\left(3+x\right)^2} = 1,$$

where $x \equiv a_1 \cdot a_2$. For this equation to be satisfied, the coefficients of similar power of x in the numerator and denominator must equal each other, and this leads to the three equations,

$$n_1 + 2n_3 + n_4 = 9 \ , \ n_2 + n_3 = 3 \ \text{ and } \ n_4 + n_5 = 1 \qquad (2.12)$$

II. If $b_1 = b_2 = -a_1$, then P=0 (see (2.2)), and this gives the equation

$$P = \frac{n_1 + n_2 - 2n_3 + (n_2 - 2n_3 + 2n_4 + n_5)x}{4\left(3+x\right)} = 0$$

The coefficients of each power of x in the numerator must be equal to zero, and this leads to the pair of equations

$$n_1 + n_2 - 2n_3 = 0 \ \text{ and } \ n_2 - 2n_3 + 2n_4 + n_5 = 0 \qquad (2.13)$$

The five equations in (2.12) and (2.13) can be solved to give

$$n_1 = 3, n_2 = 1, n_3 = 2, n_4 = 2, n_5 = -1 \qquad (2.14)$$

Substituting these back into (2.9), the squared overlap becomes

$$P \equiv \left|\left\langle b_1, b_2 \middle| a_1, a_2 \right\rangle\right|^2 = \frac{\begin{bmatrix} 4 + 2\left(a_1 + a_2\right) \cdot \left(b_1 + b_2\right) - \left(1 - a_1 \cdot a_2\right)\left(1 - b_1 \cdot b_2\right) \\ + 2\left\{\left(a_1 \cdot b_1\right)\left(a_2 \cdot b_2\right) + \left(a_1 \cdot b_2\right)\left(a_2 \cdot b_1\right)\right\} \end{bmatrix}}{\left(3 + a_1 \cdot a_2\right)\left(3 + b_1 \cdot b_2\right)} \tag{2.15}$$

By comparison, the result for a spin-1/2 particle is

$$\left|\left\langle a \middle| b \right\rangle\right|^2 = \frac{1}{2}\left(1 + a \cdot b\right) \tag{2.16}$$

The spin-1 state has much more complicated overlap. We use (2.15) as the starting point for the derivation of many of the new results of this project.

(c) Orthogonalities between spin-1 States

We wish to explore all the types of orthogonalities that are possible between C-, A- and D- states, both with each other and among themselves. All our results follow from examining (2.15) in a number of limiting cases. We state our results in the form of several propositions.

Proposition 2.1 Two C- states are orthogonal if their vectors point in opposite directions.

Proof: Take $a_1 = a_2 = a$ and $b_1 = b_2 = b$ in (2.15). Then

$$\left|\langle b,b|a,a\rangle\right|^2 = \left|\langle b|a\rangle\right|^4 = \left(\frac{1+a\cdot b}{2}\right)^2 , \qquad (2.17)$$

from which one sees that states are orthogonal only if $a = -b$, as stated.

<u>Remark</u>: Note that (2.17) is the square of (2.16), showing that the squared overlap of two C-states is smaller than the overlap of the corresponding spin-1/2 states. This effect only becomes more pronounced as we go to the C-states of higher spin systems (i.e. states with all their Majorana vectors pointing in the same direction).

Proposition 2.2 Two A-states are orthogonal only if their axes are orthogonal.

Proof: Take $a_1 = -a_2 = a$ and $b_1 = -b_2 = b$ in (2.15) to find that

$$\left|\langle b,-b|a,-a\rangle\right|^2 = \left(a\cdot b\right)^2 , \qquad (2.18)$$

from which it follows that the states are orthogonal only if $a\cdot b = 0$, or their axes are orthogonal.

Proposition 2.3 A C-state and an A-state are orthogonal only if they have a common axis.

Proof: On taking $a_1 = a_2 = a$ and $b_1 = -b_2 = b$ in (2.15) one finds that

$$\left| \langle b,-b | a,a \rangle \right|^2 = \frac{1}{2}\left[1 - \left(a \cdot b \right)^2 \right],$$ (2.19)

which vanishes only if $a.b = \pm 1$, or the states have the same axis.

Proposition 2.4 A C-state can be orthogonal to a D-state only if its vectors are the negatives of one of the vectors of D-state

Proof: On taking $b_1 = b_2 = z$ in (2.15) one finds that

$$\left| \langle z,z | a_1,a_2 \rangle \right|^2 = \frac{\left(1 + z \cdot a_1 \right)\left(1 + z \cdot a_2 \right)}{3 + a_1 \cdot a_2}$$ (2.20)

This expression vanishes only if $z = -a_1$ or $-a_2$, as stated. <u>Remark:</u> One can ask which C-State has the maximum overlap with a given D-state. An analysis of (2.20) shows that the C-state whose axis coincides that of the D-state has maximum overlap with it provided its vectors make an acute angle with the vectors of the D-state; on the other hand, if the angle is obtuse, the overlap is a saddle point. This is shown in the table below, where the first two C-states are the ones whose axes coincide with the D-state, whereas the last two C-states are the ones orthogonal to the D-state.

C-State	Squared overlap	Nature of overlap
$\left\|\hat{n},\hat{n}\right\rangle,\ \hat{n}=\dfrac{a_1+a_2}{\sqrt{2(a_1\cdot a_2)}}$	$\dfrac{\left(1+\hat{n}\cdot a_1\right)^2}{3+a_1\cdot a_2}$	Absolute maximum
$\left\|-\hat{n},-\hat{n}\right\rangle,\ \hat{n}=\dfrac{a_1+a_2}{\sqrt{2(1+a_1\cdot a_2)}}$	$\dfrac{\left(1-\hat{n}\cdot a_1\right)^2}{3+a_1\cdot a_2}$	Local Maximum
$\left\|-a_1,-a_1\right\rangle$	0	Orthogonal
$\left\|-a_2,-a_2\right\rangle$	0	Orthogonal

Table 1: Squared Overlap of C-state vectors of different orientation with the D-state $\left|a_1,a_2\right\rangle$.

Illustration: Consider a D-state with vectors lying in the x-z plane and equally inclined to the z-axis at an angle of 70^0 i.e. $a_1=\left\langle\sin(70^0),0,\cos(70^0)\right\rangle$, $a_2=\left\langle-\sin(70^0),0,\cos(70^0)\right\rangle$ and an arbitrary C-state with vector $\hat{n}=\left\langle\sin\theta,0,\cos\theta\right\rangle$ in the same plane. The squared overlap of these C and D states is an absolute maximum when the axis of C-state coincides with that of D-state i.e. when $\hat{n}=\dfrac{a_1+a_2}{\sqrt{2(a_1\cdot a_2)}}=\left\langle 0,0,1\right\rangle$ (which occurs at θ =0) and the squared overlap is a local maximum when the C-state axis is opposite to that of the D-state i.e. $\hat{n}=-\dfrac{a_1+a_2}{\sqrt{2(a_1\cdot a_2)}}=\left\langle 0,0,-1\right\rangle$ (i.e. $\hat{n}$ at $\theta=\pi$). Also the squared overlap is zero when the C-state axis lies along $-a_1$ or $-a_2$ (i.e. $\hat{n}$ at $\theta=\pi+\dfrac{7\pi}{18}$ or $\theta=\pi-\dfrac{7\pi}{18}$).

These limiting cases can be illustrated by plotting the squared overlap of the D-state $\left|a_1,a_2\right\rangle$ with the C-states $\hat{n}=\left\langle\sin\theta,0,\cos\theta\right\rangle$ for variable θ as shown in Figure 3.

0.7

0.6

0.5

0.4

Squared
Overlap

0.3

0.2

0.1

0

0    $\dfrac{\pi}{2}$ π $\dfrac{3\pi}{2}$ 2π

$\theta\,(radians)$
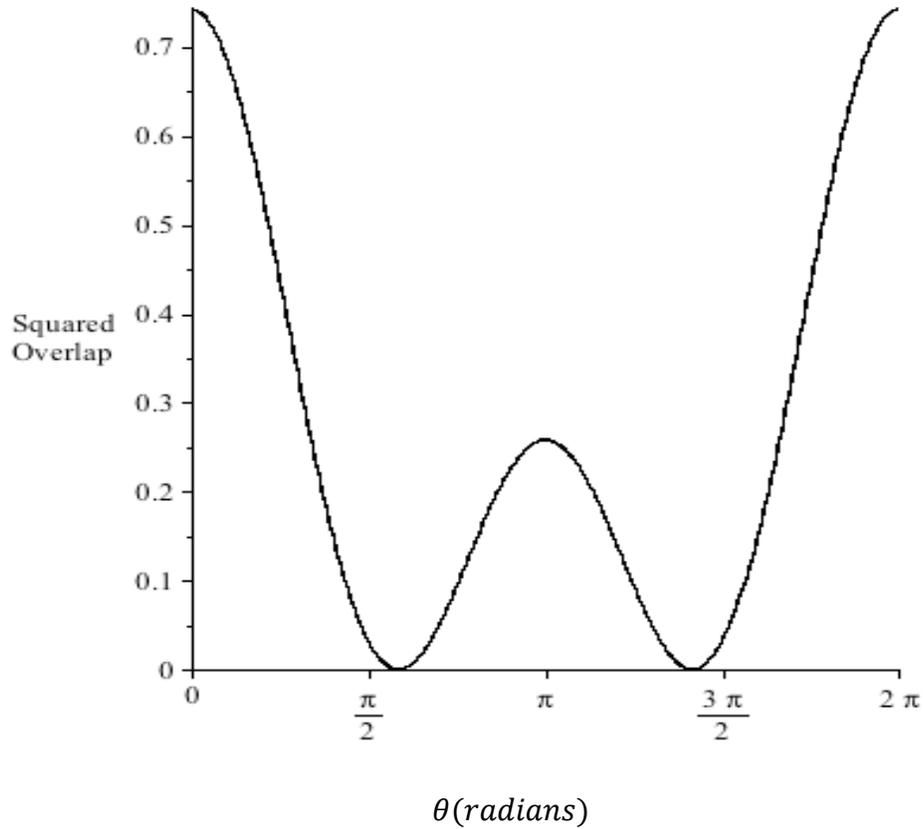
Figure 3: Squared Overlap of D-state with various C-states. Note the squared overlap is absolute maximum at θ=0 and 2pi, minimum at θ=0.61pi and 1.38pi and local maximum at θ=pi

Proposition 2.5 An A-state can be orthogonal to a D-state only if the two have a common axis.

Proof: Using $b_1 = -b_2 = z$ in (2.15) one finds that

$$\left|\left\langle z,-z\middle|a_1,a_2\right\rangle\right|^2 = \frac{1-2\left(z\cdot a_1\right)\left(z\cdot a_2\right)+a_1\cdot a_2}{3+a_1\cdot a_2} \qquad (2.21)$$

The numerator is quadratic in z that goes to zero only if

$$z = \pm\frac{a_1+a_2}{\sqrt{2\left(1+a_1\cdot a_2\right)}} \qquad (2.22)$$

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showing that the axis of A-state coincides with that of the D-state, as claimed. <u>Remark:</u>

The table below shows the squared overlaps of some A-states with the D-state $\left|a_1, a_2\right\rangle$.

The vectors a_1 and a_2 are taken to lie in the x-z plane, with the z-axis bisecting them, and x and y denote unit vectors along the x- and y- axes.

A-State	Squared overlap	Nature of overlap
$\left|x, -x\right\rangle$	$\dfrac{2}{3 + a_1 \cdot a_2}$	Absolute maximum
$\left|y, -y\right\rangle$	$\dfrac{1 + a_1 \cdot a_2}{3 + a_1 \cdot a_2}$	Saddle Point
$\left|a_1, -a_1\right\rangle, \left|a_2, -a_2\right\rangle$	$\dfrac{1 - a_1 \cdot a_2}{3 + a_1 \cdot a_2}$	Some value
$\left|z, -z\right\rangle$	0	Orthogonal

Table 2: Squared Overlap of D-state $\left|a_1, a_2\right\rangle$ lying in x-z plane with various A- states

<u>Illustration:</u>

Consider D-state vectors in the x-z plane bisected by the z-axis at an angle $\dfrac{\pi}{6}$ so that

$a_1 = \left\langle \dfrac{1}{2}, 0, \dfrac{\sqrt{3}}{2} \right\rangle$ and $a_2 = \left\langle -\dfrac{1}{2}, 0, \dfrac{\sqrt{3}}{2} \right\rangle$. Also consider arbitrary A-state vectors

$b_1 = \left\langle \sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta \right\rangle$ and $b_2 = \left\langle \sin(\pi + \theta)\cos\varphi, \sin(\pi + \theta)\sin\varphi, \cos\theta \right\rangle$. The

squared overlap of these two states is absolute maximum when the A-state reduces to

$\left|x, -x\right\rangle$ i.e. when $(\theta, \varphi) = \left(\dfrac{\pi}{2}, n\pi \right)$ where $n = 0, 1, 2$. And the squared overlap is local

maximum when A-state reduces to $\left|y, -y\right\rangle$ i.e. when $(\theta, \varphi) = \left(\dfrac{\pi}{2}, \dfrac{n\pi}{2} \right)$ where $n = 1, 3$. It is

zero when A-state is $\left|z,-z\right\rangle$ i.e. when $(\theta,\varphi)=(0,0)$. These limiting cases are shown in the Figure 4 shown below.

Squared OVerlap of D-and various A-states
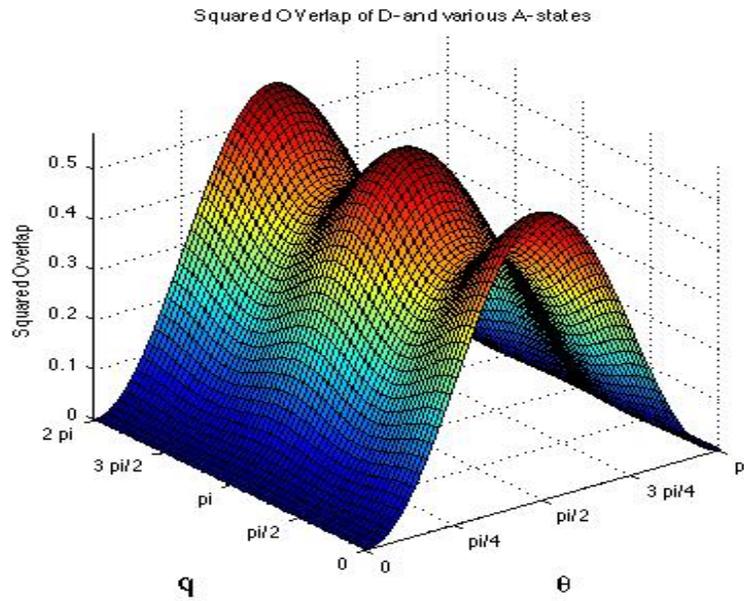
Figure 4: Squared Overlap of given D- state $\left|a_1,a_2\right\rangle$ and A-state $\left|b_1,b_2\right\rangle$ varying with angle θ and φ. Note that the squared overlap is absolute maximum at $(\theta,\varphi) = \left(\frac{\pi}{2},0\right)$, $\left(\frac{\pi}{2},\pi\right)$ and $\left(\frac{\pi}{2},2\pi\right)$, saddle points at $(\theta,\varphi) = \left(\frac{\pi}{2},\frac{\pi}{2}\right)$ and $(\theta,\varphi) = \left(\frac{\pi}{2},\frac{3\pi}{2}\right)$ and zero at $(\theta,\varphi) = (0,0)$.

Proposition 2.6 Two D-states $\left|a_1,a_2\right\rangle$ and $\left|b_1,b_2\right\rangle$ are orthogonal only if b_2 is related to the remaining vectors by

$$b_2 = -\left[\frac{2\left(p_2+1\right)a_1 + 2\left(p_1+1\right)a_2 + \left(1-p_3\right)b_1}{2\left(p_1+p_2\right)+p_3+3}\right] \tag{2.22}$$

where $p_1 = a_1 \cdot b_1, p_2 = a_2 \cdot b_1$ and $p_3 = a_1 \cdot a_2$.

Proof:

If $\left|b_1,b_2\right\rangle$ is to be orthogonal to $\left|a_1,a_2\right\rangle$, the numerator of (2.15) must vanish. The numerator is linear in b_2, and the condition that it vanishes can be put, after a bit of algebra, in the form $u \cdot b_2 = 1$, where u is a unit vector identical to the expression given on the right side of (2.22). The condition $u \cdot b_2 = 1$ can therefore be satisfied by taking $b_2 = u$, which is what we wanted to prove.

Remark:

This criterion actually applies to an arbitrary pair of spin-1 states, and not just D-states. An alternative criterion for the orthogonality, based on Mobius transformations, will be given in a later section. Penrose has given yet another criterion that makes use of stereographic projection.

Fig. 5 and Fig. 6 show, in pictorial form, a pair of orthogonal states of each of the types discussed in Proposition 2.1-2.5. In a later section we will give a geometrical construction of a D-state orthogonal to another D-state.

(a) C-C Orthogonal State

  Bold small arrows represent

Coherent states
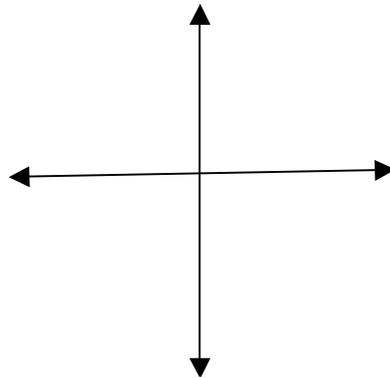
(b) A-A Orthogonal State

Each of the lines represent A-state

Figure 5:Orthogonalities of Spin-1 States

(c) C-A Orthogonal State

Bold small arrow is C-state

Double-headed arrow is A-state
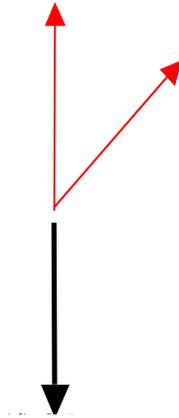
(d) C-D Orthogonal State

A pair of red arrows-D-state

Black small arrow is C-state
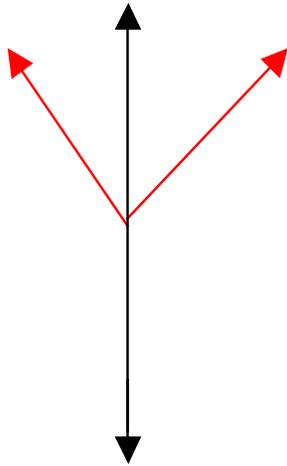
(e) A-D Orthogonal State
A pair of red arrows is D-state
Double headed arrow is A-state

Figure 6: Orthogonalities of Spin-1 States

(d) Possible Bases of spin-1 States

In this section we wish to construct triads of mutually orthogonal states, or bases. From a purely combinatorial viewpoint, the C, A and D states can form 10 types of bases: CCC, CCA, CAA, AAA, CCD, CDD, AAD, ADD, ACD, and DDD. However we will show that Propositions 2.1-2.6 rule out the possibilities CCC, CAA, CCD, AAD and ACD (the ordering of letters within a basis is not significant) and allow only the five others.

An easy way to construct the bases that exist and rule out the ones that do not is

- Add a third state to each of the orthogonal pairs of states shown in Fig.5.

- Next impose the constraints dictated by Proposition 2.1-2.6

- Rule out the bases that do not exist

(d.1) CAC Basis:

If we start from Fig.5 (a), we see from Proposition 2.3 that it is possible to add an A-state with the same axes as the two C-states present to get a CAC basis, shown in Fig.7. Since a pair of orthogonal states uniquely fix the third member of the basis, the existence of CAC basis rules out the existence of any basis that differs from it by a single letter, thus the CCC, CDC, CAA and CAD bases all get ruled off in a single stroke.
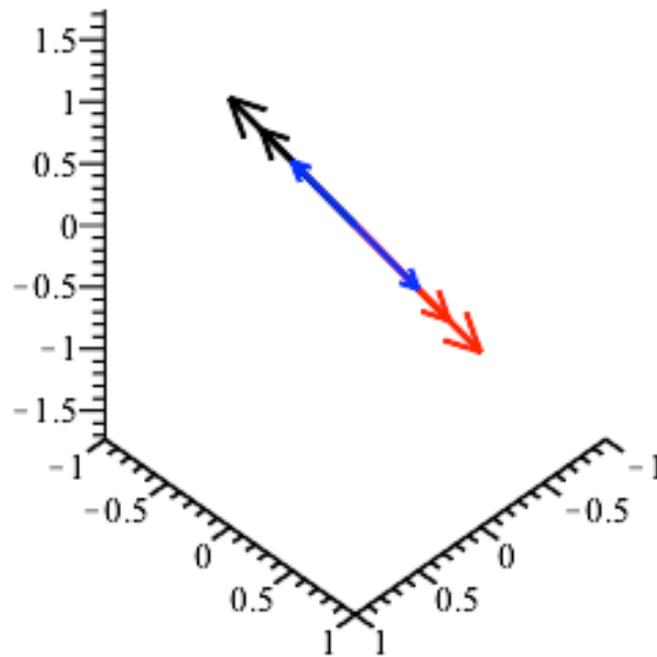
Figure 7: CAC Basis. Here black arrows and red arrows represent C-states and the double-headed blue arrow is A-state

(d.2) AAA Basis:

Next let us look at Fig.5 (b). From Proposition 2.2 one can add a third A-state whose axis is orthogonal to the axes of the two A-states already present to get an AAA basis, shown in Fig.8. The existence of an AAA basis rules out the existence of an AAD basis.
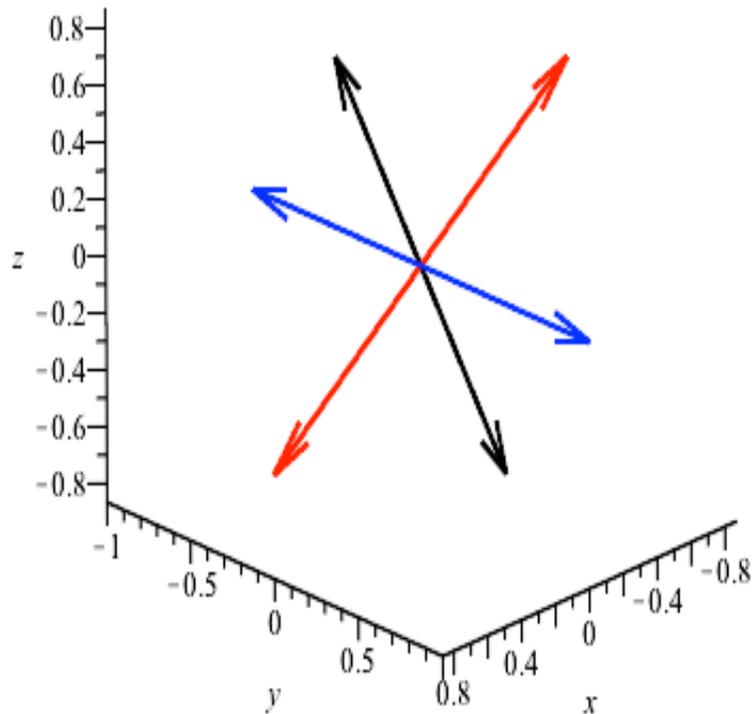
Figure 8: AAA Basis. Each of the colored double-headed arrows represents a distinct A-state.

It remains to consider the cases CDD, ADD and DDD. All these types of bases do exist, and we give constructions for them in the next three sections.

(d.3) The CDD basis

We begin with the orthogonal C- and D- states shown in Fig.6 (d). We take the D-state to be $\left| a_1, a_2 \right\rangle$ and the C-state to be $\left| -a_1, -a_1 \right\rangle$. By Proposition 2.4, the D- state that completes the basis must have one of its vectors be the negative of the C-state vectors i.e. the new D-state should be of the form $\left| a_1, b_2 \right\rangle$, where it remains to determine b_2

Two D-states are however required to be orthogonal. Proposition 2.6 dictates that the two D-states are orthogonal if and only if

$$b_2 = -\left[\frac{\left(3 + a_1 \cdot a_2\right)a_1 + 4a_2}{5 + 3a_1 \cdot a_2} \right]$$

(2.23)

This basis is depicted in Fig. 9. Note that the vector b_2 lies in the same plane as a_1 and a_2. And it has a negative component along each of them. As the vector a_2 tends towards a_1, one of the D-states goes into a C-state and the other into an A-state, and the basis passes over into CAC basis.
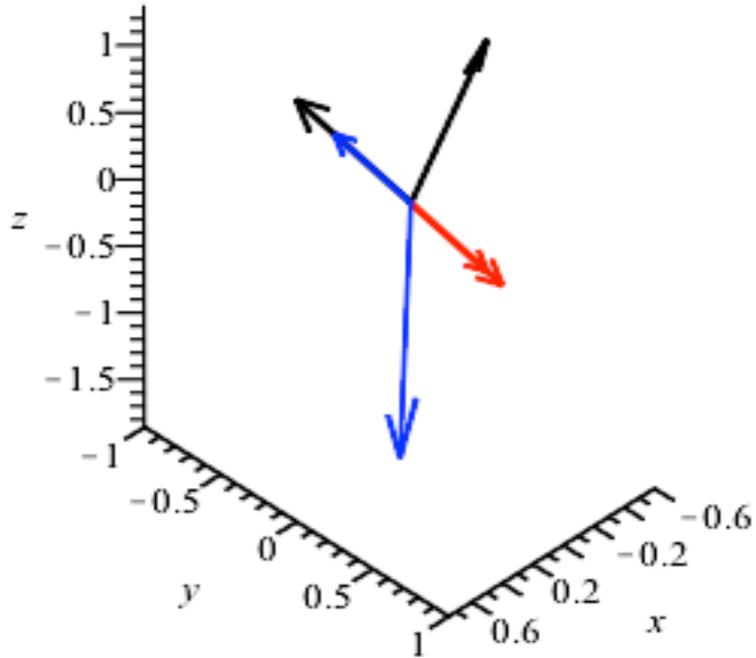
Figure 9: CDD Basis (Here a pair of black arrows represents D-state $\left|a_1, a_2\right\rangle$, a pair of blue arrows represents D-state $\left|a_1, b_2\right\rangle$ and a pair of red arrows represents C-state $\left|-a_1, -a_2\right\rangle$. Note that b_2 lies in the plane formed by $-a_1$ and $-a_2$)

(d.4) The ADD basis

Let us begin with the A-state $|z,-z\rangle$, where z is the unit vector along the z-axis. We know from Proposition 2.5 that any D-state orthogonal to this A-state must have both its vectors equally inclined to the z-axis and lying in a plane that passes through the z-axis. Thus we can take these D-states to be of the form $|a_1,a_2\rangle$ and $|b_1,b_2\rangle$, where

$a_1 = (\sin\theta_0, 0, \cos\theta_0)$, $a_2 = (-\sin\theta_0, 0, \cos\theta_0)$, $b_1 = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ and

$b_2 = (-\sin\theta\cos\phi, -\sin\theta\sin\phi, \cos\theta)$. It remains to fix the values of θ and ϕ in terms of θ_0 by requiring that the states $|a_1,a_2\rangle$ and $|b_1,b_2\rangle$ are orthogonal to each other. Using (2.22) to impose the orthogonalities leads to the three equations:

$$(\cos\theta + \cos\theta_0)(1 + \cos\theta_0\cos\theta) = 0 \tag{2.24}$$

$$-\sin^2\theta_0 \sin\theta\cos\phi = (2\cos\theta_0\cos\theta + \cos^2\theta_0 + 1)\sin\theta\cos\phi \tag{2.25}$$

$$\sin^2\theta_0 \sin\theta\sin\phi = (2\cos\theta_0\cos\theta + \cos^2\theta_0 + 1)\sin\theta\sin\phi \tag{2.26}$$

whose unique solution is $\theta = \pi - \theta_0$ and $\phi = \pi/2$. This basis is shown in Fig.10. If $\theta = \pi/2$, the basis degenerates into an AAA basis.

Illustration: Consider D-state vectors (black arrows) inclined at an equal angle $\theta = \dfrac{\pi}{6}$ to

the z-axis in the x-z plane as shown in Fig. 10. Then our argument says that A- state

vectors orthogonal to this D-state must lie along the z-axis as given by blue

arrows. The third D-state that is orthogonal to the above A-D states must have its state

vectors inclined at an angle $\theta = \pi - \theta_0 = \dfrac{5\pi}{6}$ and rotated about the z-axis by an angle of

$\varphi = \dfrac{\pi}{2}$ as given by red arrows.
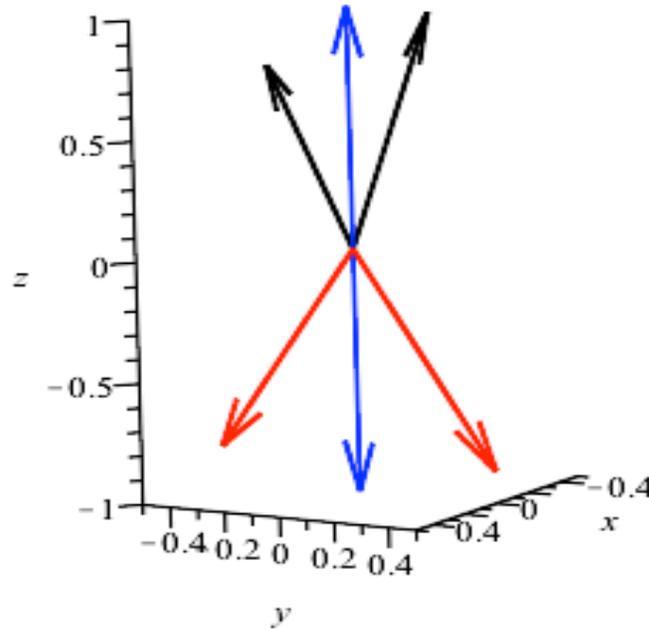
Figure 10: ADD Basis (Here a pair of black arrows in the X-Z plane with an angle of $\dfrac{\pi}{6}$

with z-axis represents the first D-state, the double headed blue arrow along the Z-axis

represents A-state and a pair of red arrow rotated about Z-axis by $\dfrac{\pi}{2}$ with an angle of $5\dfrac{\pi}{6}$

with Z-axis represents the second D-state)

(d.5) The Conical DDD basis

The construction of a general DDD basis is quite involved. However there is a special type of DDD basis that is of considerable interest in its on right: it is one in which angle between the state vectors is the same of for all the states in the basis. We will refer to this special type of basis as a "conical" basis, for a reason that will soon become clear.

Let 1,2 and 3 be the states of the conical basis. The rotation that takes the vectors of state 1 into those of state 2 will also take the vectors of 2 into those of a state 3' that is orthogonal to 2. However state 3' must coincide with state 3, since the state that completes an orthogonal triad with 1 and 2 must be unique.[1] The same rotation, if performed on state 3, must take it into state 1, and this shows that the rotation must be a 3-fold rotation about some axis. In other words, the vectors of the states 1,2 and 3 must lie on a double cone and be cycled into each other by rotations of $120°$ about the axis of the double cone. Because all the vectors lie on a double cone, we refer to this basis as a conical basis.

Let us construct the states of a conical basis whose vectors lie on a double cone of half angle θ centered on the z-axis. We choose the first state of the basis to be $\left|a_1,a_2\right\rangle$, with $a_1 = \left(\sin\theta,0,\cos\theta\right)$ and $a_2 = \left(\sin\theta\cos\phi,\sin\theta\sin\phi,-\cos\theta\right)^2$. The other two states of the basis can be obtained from $\left|a_1,a_2\right\rangle$ by rotating its vectors by $2\pi/3$ and $4\pi/3$ about the z-axis. If we denote these states by $\left|b_1,b_2\right\rangle$ and $\left|c_1,c_2\right\rangle$, the vectors associated with them are

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[1] The state 3' cannot coincide with 1 since the rotation from 1 to 2 would then be a 2-fold rotation and incapable of including a third orthogonal state in its orbit.
[2] Note that we have chosen one vector to lie on the upper half of the double cone and the other on the lower half. One can verify, by a calculation similar to that given here, that choosing both vectors to lie on the same half of the double cone will not allow the construction to succeed.

$$b_1 = \left(\sin\theta\cos\frac{2\pi}{3}, \sin\theta\sin\frac{2\pi}{3}, \cos\theta\right), \quad b_2 = \left(\sin\theta\cos\left(\phi+\frac{2\pi}{3}\right), \sin\theta\sin\left(\phi+\frac{2\pi}{3}\right), -\cos\theta\right)$$

$$c_1 = \left(\sin\theta\cos\frac{4\pi}{3}, \sin\theta\sin\frac{4\pi}{3}, \cos\theta\right), \quad c_2 = \left(\sin\theta\cos\left(\phi+\frac{4\pi}{3}\right), \sin\theta\sin\left(\phi+\frac{4\pi}{3}\right), -\cos\theta\right)$$

We can now fix the value of ϕ by requiring that $\left|b_1,b_2\right\rangle$ be orthogonal to $\left|a_1,a_2\right\rangle$. Using (2.15) to impose the orthogonality and doing some algebra shows that ϕ is determined in terms of θ by

$$\cos\phi = -2\cot^2\theta \tag{2.27}$$

The angle, Ω, between the vectors in any state of this basis is given by

$$\Omega = \cos^{-1}\left(-3\cos^2\theta\right) \tag{2.28}$$

It is clear from (2.27) that conical bases can exist only if $\sqrt{2} \leq \tan\theta < \infty$ and that the

possible values of ϕ lie in the range $\frac{\pi}{2} \leq \phi < \pi$. The limiting cases of these bases are worth noting:

(a) If $\theta = \arctan\left(\sqrt{2}\right) = 54.74°$, then $\phi = \Omega = \pi$ and the conical basis degenerates into an AAA basis whose vectors are all equally inclined to the z-axis.

(b) If $\theta = \arctan\left(\infty\right) = 90°$, then $\phi = \Omega = \pi/2$ and the vectors reduce to

$$\vec{a}_1 = \left(1,0,0\right) \quad , \quad \vec{a}_2 = \left(0,1,0\right) \tag{2.29}$$

$$\vec{b}_1 = \left(-1/2, \sqrt{3}/2, 0\right) \quad , \quad \vec{b}_2 = \left(-\sqrt{3}/2, -1/2, 0\right) \tag{2.30}$$

$$\vec{c}_1 = \left(-1/2, -\sqrt{3}/2, 0\right) \quad , \quad \vec{c}_2 = \left(\sqrt{3}/2, -1/2, 0\right) \tag{2.31}$$

The vectors all lie in the x-y plane, and the vectors of each state are at right angles to each other. In fact, the vectors of the three states point in the directions $\left(0°, 90°\right)$, $\left(120°, 210°\right)$ and $\left(240°, 330°\right)$.

From (2.28) it follows that the angle between the vectors of a conical basis state must be obtuse (or $90°$, at the very least). D-states whose vectors make an acute angle with each other can never be part of a conical basis.

Illustration: In the Fig.11 given below, a double cone has half angle $\theta = \dfrac{\pi}{3}$ and from (2.28), $\phi = \arccos\left(-2\cot^2\theta\right) = 2.3rad = 131.81°$ Substituting these values into above D-state vectors used in the construction of DDD basis we get the figure 8. D-state $|a_1, a_2\rangle$, $|b_1, b_2\rangle$ and $|c_1, c_2\rangle$ are represented by a pair of black, red and blue arrows respectively. The angle Ω between the vectors of each D-state is 2.418 rad or $138.58°$ as given by (2.28).
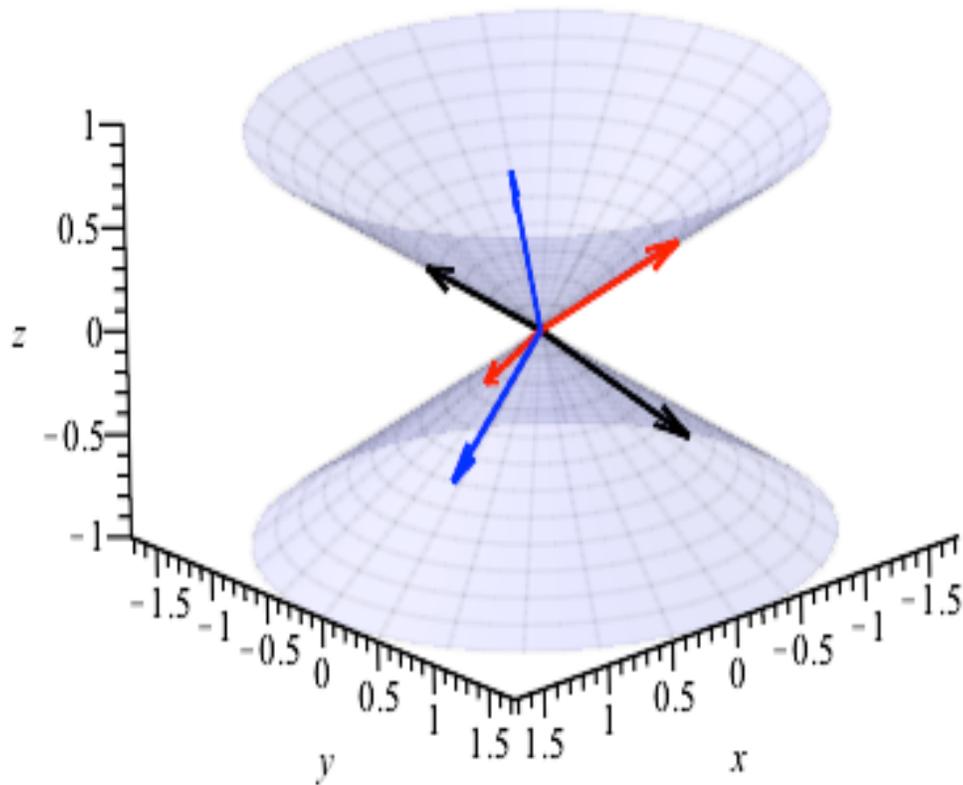
Figure 11: DDD Conical Basis (Here the conical basis lies on the two branches double-cone of half angle $\frac{\pi}{3}$. Angle between each pair of vectors of D-state is obtuse ($\mathbf{138.58^0}$) and one vector of the state lies on the upper half cone and another lies on the lower half cone.)

(f) Construction of a general DDD basis

We discuss the construction of a general DDD basis i.e. one whose members are not constrained to have a constant angle between their vectors. We carry out the discussion in terms of the complex parameters of the states, since the algebra is then simpler, as is the associated geometrical interpretation of the results.

Consider the D-state $\left| \alpha_1, \alpha_2 \right\rangle$, where we refer to the state by its complex parameters. From (2.3) we see that we can construct state $\left| \beta_1, \beta_2 \right\rangle$ orthogonal to $\left| \alpha_1, \alpha_2 \right\rangle$ by taking β_2 to be given in terms of α_1, α_2 and β_1 by

$$\beta_2 = -\left[\frac{1 + \frac{1}{2}\left(\alpha_1 + \alpha_2\right)^* \beta_1}{\frac{1}{2}\left(\alpha_1 + \alpha_2\right)^* + \left(\alpha_1 \alpha_2\right)^* \beta_1} \right] \tag{2.32}$$

Eq. (2.32) is an example of a Mobius transformation, which generally maps a complex number z into another complex number z' according to the equation

$$z' = \frac{az + b}{cz + d} \tag{2.33}$$

where a, b, c and d are complex numbers such that $ad - bc \neq 0$. The inverse of the Mobius transformation (2.33) is the second Mobius transformation

$$z' = \frac{dz - b}{-cz + a} \tag{2.34}$$

One sees that (2.32) is an involuntary Mobius transformation (i.e. one that is its own inverse) that maps the complex parameters of any state into those of another state that is orthogonal to it.

We next construct the state $|\gamma_1,\gamma_2\rangle$ that is orthogonal to both $|\alpha_1,\alpha_2\rangle$ and $|\beta_1,\beta_2\rangle$ and therefore forms a basis with them. Clearly, γ_1 and γ_2 must be the unique complex numbers that are mapped into each other by the pair of involuntary Mobius transformation defined by the states $|\alpha_1,\alpha_2\rangle$ and $|\beta_1,\beta_2\rangle$. From (2.32) we see that γ_1 and γ_2 are the two solutions of the quadratic equation for the complex variable z:

$$\frac{1+\frac{1}{2}\left(\alpha_1+\alpha_2\right)^* z}{\frac{1}{2}\left(\alpha_1+\alpha_2\right)^*+\alpha_1\alpha_2 z} = \frac{1+\frac{1}{2}\left(\beta_1+\beta_2\right)^* z}{\frac{1}{2}\left(\beta_1+\beta_2\right)^*+\beta_1\beta_2 z} \qquad (2.35)$$

Given three arbitrary complex numbers α_1,α_2 and β_1, we can use (2.32) and (2.35) to calculate the numbers β_2,γ_1 and γ_2 that, together with the first three numbers, define a basis of three mutually orthogonal states.

We give some concrete examples of the above construction taking $\alpha_1=ik$ and $\alpha_2=-ik$, where k is a positive real number[3]. From (2.32) we see that $\beta_2=-\left(k^2\beta_1\right)^{-1}$ and from (2.35) we find that the parameters of the third orthogonal state are

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[3] This choice is not as restrictive as it might seem, since any D-state can always be brought into this form by a suitable Mobius transformation.

$$\gamma_{1,2} = \frac{-\left(1+\dfrac{1}{k^4}\right) \pm \sqrt{\left(1+\dfrac{1}{k^4}\right)^2 + \dfrac{1}{k^2}\left(\beta_1 - \dfrac{1}{k^2\beta_1}\right)^{*2}}}{\left(\beta_1 - \dfrac{1}{k^2\beta_1}\right)^*} \qquad (2.36)$$

The table below shows the bases that result if we always take $\alpha_{1,2} = \pm ik$ but make various choices for β_1.

(α_1, α_2)	(β_1, β_2)	(γ_1, γ_2)	Comments
$(ik, -ik)$	$(0, \infty)$	$\left(\dfrac{1}{k}, -\dfrac{1}{k}\right)$	DAD basis
$(ik, -ik)$	$\left(ik, \dfrac{i}{k^3}\right)$	$\left(-\dfrac{i}{k}, -\dfrac{i}{k}\right)$	DDC basis
$(ik, -ik)$	$\left(-ik, -\dfrac{i}{k^3}\right)$	$\left(\dfrac{i}{k}, \dfrac{i}{k}\right)$	DDC basis

Table 3: Different Bases formed by the same D-state vectors in the first column and various state vectors in column 2 and 3 resulted from three different choices of $\boldsymbol{\beta_1}$.

The types of bases that result in each case are listed in the last column. As $k \to 1$, all the above bases go over into AAA bases. For more general choices of β_1 we would end up with DDD bases with no two states having the same angle between their vectors.

A geometrical interpretation of the above construction can be given by plotting the parameters of the states in the complex plane, as in Fig 12. Consider $k = 0.8$. Then $\alpha_1 = i0.8$ and $\alpha_2 = -i0.8$ can be represented by the two blue arrows on the imaginary axis. Once β_1 has been picked say $\beta_1 = 2 + 2i$ (longer red arrow in the figure), β_2 can be obtained by inverting β_1 in a circle of radius $1/k$ centered at the origin and then performing a reflection in the imaginary axis. Doing so gives us $\beta_2 = -0.3906 + 0.3906i$ as represented by the shorter red arrow. The construction for the points γ_1 and γ_2 is a little more involved.

Let $C = \dfrac{1}{2}\left(\beta_1 - \dfrac{1}{k^2 \beta_1}\right)^*$, $A = -k^2 C$ and $B = -\dfrac{1}{k^2} - k^2$. Shift the origin of the complex plane to the point $z_0 = -\dfrac{B}{2A}$ and let $z' = z - z_0$ be the location of an arbitrary point relative to this new origin. If $z' = x' + iy'$, the points γ_1 and γ_2 are determined as the intersections of the rectangular hyperbolas

$$x'^2 - y'^2 = \mathrm{Re}\left(\frac{B^2 - 4AC}{4A^2}\right) \quad \text{and} \quad 2x'y' = \mathrm{Im}\left(\frac{B^2 - 4AC}{4A^2}\right) \tag{2.37}$$

whose symmetry axes are rotated $45°$ relative to each other as shown in the figure. Each hyperbola has two branches, with each branch of one intersecting a branch of other to give a solution. The solutions are in fact the inverses of each other in the point z_0, and so one is easily obtained from the other.

Yet another way of generating one solution from another is to use the same procedure that was used to obtain β_2 from β_1 (namely, inversion in a circle of radius $1/k$ centered at the origin, followed by reflection in the imaginary axis). Doing so gives us the γ_1 and γ_2 (black arrows) exactly at the intersections of the rectangular hyperbolas as shown in the figure. It is interesting that γ_1 and γ_2 can be obtained from each other by these rather different methods.
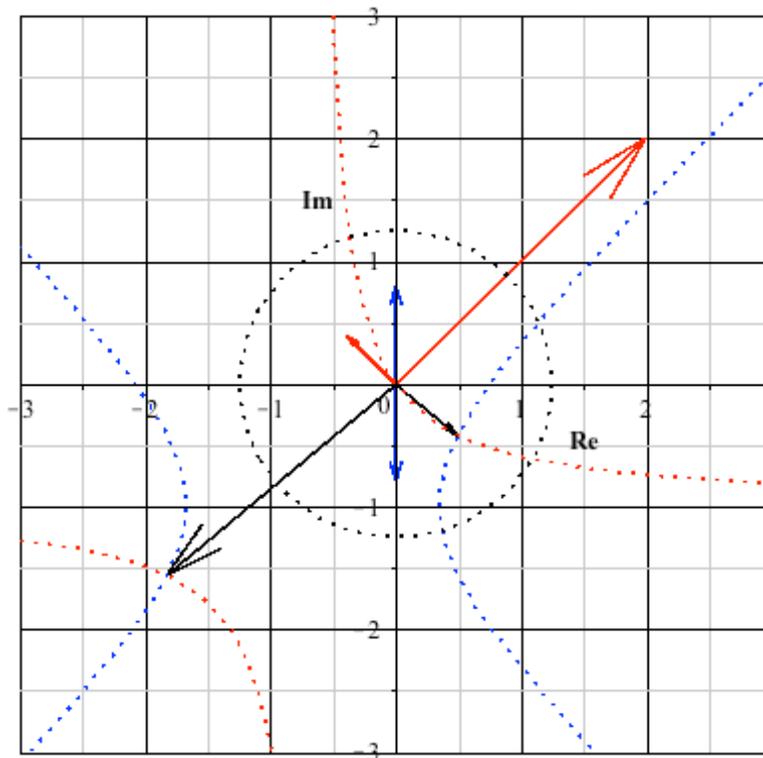
Figure 12: Construction of general DDD basis. Here $\alpha_1 = 0.8i$ and $\alpha_1 = -0.8i$ are represented by blue arrows. $\beta_1 = 2 + 2i$ and $\beta_2 = -0.396 + 0.396i$ are longer and smaller red arrow respectively. The blue dotted hyperbola and the red doted hyperbola intersect to give γ_1 *and* γ_2 and the black arrows also formed by inversion in a circle point at the same γ_1 *and* γ_2 .

(g) Unitary Transformation between spin-1 states

The various bases we have worked out in the earlier sections allow us to construct unitary transformations that we will take any spin-1 state into any other state. If we wish to transform the state $|1\rangle$ into the state $|1'\rangle$, all we have to do is to identify a basis to which $|1\rangle$ belongs (say, $|1\rangle, |2\rangle, |3\rangle$) and one to which $|1'\rangle$ belong (say, $|1'\rangle, |2'\rangle, |3'\rangle$) and set up the transformation as $U = |1'\rangle\langle 1| + |2'\rangle\langle 2| + |3'\rangle\langle 3|$. Of course there are many ways in which this can be done, since the other members of the two bases are not fixed. As a simple example, let us work out a transformation that takes the C-state whose vectors point up along the z-axis to the D-state whose vectors lie in the x-z plane and make an equal angles θ with the z-axis. We choose the C-state to be part of the standard basis of states with spin component +1,0 and -1 along the z-axis:

$$|1\rangle = |z,z\rangle, \quad |2\rangle = |z,-z\rangle, \quad |3\rangle = |-z,-z\rangle \tag{2.38}$$

We choose the D-state to part of a basis consisting of an A-state and another D-state, all of whose axes coincide with the z-axis. Specifically, we take the vectors of this basis to be

$$|1'\rangle = |a_1,a_2\rangle, \quad |2'\rangle = |z,-z\rangle, \quad |3'\rangle = |b_1,b_2\rangle \tag{2.39}$$

where $a_1 = \left(-\sin\theta, 0, \cos\theta\right)$, $a_2 = \left(\sin\theta, 0, \cos\theta\right)$, $b_1 = \left(0, -\sin\theta, -\cos\theta\right)$ and $b_2 = \left(0, \sin\theta, -\cos\theta\right)$. The unitary transformation that maps state $|1\rangle$, a C-state, into state $|1'\rangle$, a D-state, while also mapping $|2\rangle$ into $|2'\rangle$ and $|3\rangle$ into $|3'\rangle$, can be represented by the 3 x 3 matrix

$$U = \begin{bmatrix} \langle z,z|a_1,a_2\rangle & \langle z,z|z,-z\rangle & \langle z,z|b_1,b_2\rangle \\ \langle z,-z|a_1,a_2\rangle & \langle z,-z|z,-z\rangle & \langle z,-z|b_1,b_2\rangle \\ \langle -z,-z|a_1,a_2\rangle & \langle -z,-z|z,-z\rangle & \langle -z,-z|b_1,b_2\rangle \end{bmatrix} \qquad (2.40)$$

The elements can be worked out using (2.7) and one finds that

$$U = \frac{1}{\sqrt{2\left(1+\cos^2\theta\right)}} \begin{bmatrix} 1+\cos\theta & 0 & -1+\cos\theta \\ 0 & 1 & 0 \\ -1+\cos\theta & 0 & 1+\cos\theta \end{bmatrix} \qquad (2.41)$$

The limits $\theta = 0$ and π can be checked by setting up the matrix U directly in these cases. The matrix (2.41) does not belong to a SU (2) subgroup of SU (3) because it generally changes the angle between the Majorana vectors of an arbitrary state.

(h) Mutually Unbiased Basis

Two different spin-1 bases, $|\psi_i\rangle$ and $|\varphi_i\rangle$ with $i = 1,2,3$ are said to be mutually unbiased if $\left|\langle\psi_i|\phi_j\rangle\right|^2 = 1/3$ for all $i = 1,2,3$. For a spin-1 system, there are four mutually unbiased bases (and no more).

If $|\psi_i\rangle \equiv \left\{|a_1,a_2\rangle, |a'_1,a'_2\rangle, |a''_1,a''_2\rangle\right\}$ and $|\varphi_i\rangle \equiv \left\{|b_1,b_2\rangle, |b'_1,b'_2\rangle, |b''_1,b''_2\rangle\right\}$ represent two of the four possible mutually unbiased bases of spin-1 states, they should satisfy $\left|\langle\psi_i|\phi_j\rangle\right|^2 = 1/3$ for all $i = 1,2,3$. That is to say ,for example, $\left|\langle\psi_1|\phi_1\rangle\right|^2 =$ $\left|\langle b_1,b_2|a_1,a_2\rangle\right|^2 = 1/3$ (for $i = j = 1$) and similarly for other states in the basis. Imposing this definition of MUBs on the squared overlap formula derived in Eq. (2.5) leads the following condition that must be satisfied by each pair of mutually unbiased states:

$$3(a_1 + a_2) \cdot (b_1 + b_2) + 3\left[(a_1 \cdot b_1)(a_2 \cdot b_2) + (a_1 \cdot b_2)(a_2 \cdot b_1)\right] = 2(a_1 \cdot a_2)(b_1 \cdot b_2) \tag{2.42}$$

We now make use of the above condition to derive a basis that is mutually unbiased to a given one.Consider a state $|a_1,a_2\rangle$ with vectors $a_1 = \langle 1,0,0\rangle$ and $a_2 = \langle 0,1,0\rangle$ that lie in X-Z plane. We can rotate each of these vectors by 120 degree to obtain a state $|a'_1,a'_2\rangle$ orthogonal to the original state. Rotating each of the vectors of this new state by another 120 degree will the give a third orthogonal state $|a''_1,a''_2\rangle$. These three states are mutually orthogonal to each and together they form a basis in X-Y plane. This basis is shown in the figure 12 with each pair of vectors of the state in the basis represented by a pair of colors.

How can we get a basis that is mutually unbiased to the above basis? Let us consider a state $|b_1, b_2\rangle$ with $b_1 = \langle \sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta \rangle$ and

$b_2 = \langle \sin\theta\cos(\varphi+\varphi_0), \sin\theta\sin(\varphi+\varphi_0), -\cos\theta \rangle$. Rotating these vectors by 120 degrees and 240 about the Z-axis will give the vectors of the other two states in this basis.

If we impose the MUBs condition (2.42) we can solve for the values of φ, φ_0 and θ. Substituting these values into the state vectors of the new basis force each pair of the state vectors to lie on the surface of double cones with the given half angle θ as shown in the figure 12. The detailed derivation of the state vectors and their geometrical construction are given in the appendix B.
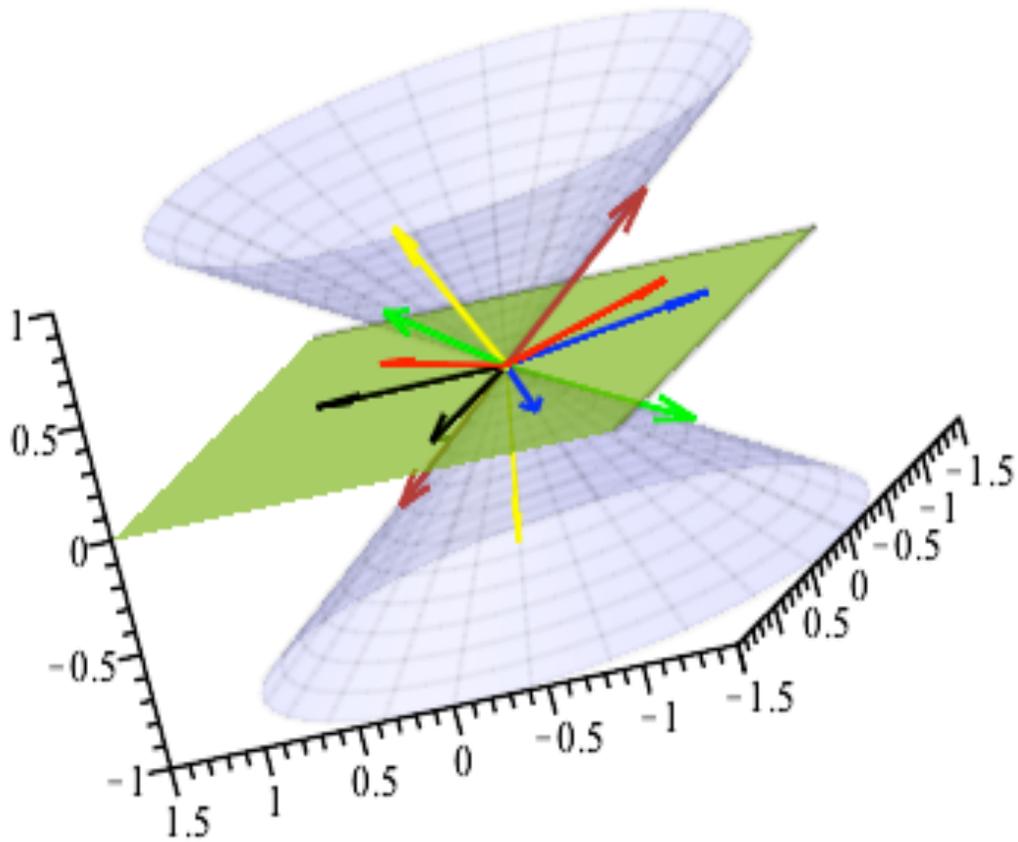
Figure 13:Two of the four possible mutually unbiased bases of a spin-1 system. Blue arrows, black arrows and red arrows lying in the green X-Y plane represent the state vectors of the first basis. Yellow arrows, brown arrows and green arrows lying on the surface of a double cone (and rotate 120 relative to each other) represent the state vectors of a second basis that is mutually unbiased to the first basis.

3. Conclusion:

We have derived the squared overlap expression for two spin-1 states in terms of their Majorana vectors. This result was used to work out all the orthogonalities and bases that are possible for three geometrically distinct types of spin-1 states. A geometrical construction is given for a set of mutually unbiased bases for a spin-1 system.

The formalism developed in this project has applications to several problems in quantum information. It can be used to discuss geometric phases in spin-1 systems in a manner that complements the earlier treatments of Hannay [4] and Arvind et al [22]. It can be used to discuss questions related to quantum contextuality in spin-1 systems [23]. And, further, it can be used to address problems related to state discrimination in spin-1 systems [24]. These are subjects for future research.

4. Appendix

Appendix A:

Consider a following state $|\psi\rangle$ expressed in the angular momentum basis.

$|\psi\rangle = |+1\rangle + |0\rangle + \lambda|-1\rangle$, ignoring an overall normalization factor

where

$$|+1\rangle = |z,z\rangle = |z_1\rangle|z_2\rangle$$

$$|0\rangle = |z,-z\rangle = \frac{1}{\sqrt{2}}\left(|z\rangle_1|-z\rangle_2 + |-z\rangle_1|z\rangle_2\right)$$

$$|-1\rangle = |-z,-z\rangle = |-z_1\rangle|-z_2\rangle$$

we can find the M-vectors (α_1, α_2) of this state by noting the relation

$$\frac{\alpha_1 + \alpha_2}{\sqrt{2}} = 1 \quad \text{and} \quad \alpha_1\alpha_2 = \lambda \tag{2.43}$$

Putting $\alpha_2 = \lambda / \alpha_1$ in the first relation transforms it into the following quadratic equation for α_1:

$$\alpha_1^2 - \sqrt{2}\alpha_1 + \lambda = 0$$

This quadratic has two complex roots, and these roots give α_1 and α_2, as you can easily check.

The expressions for the roots are:

$$\alpha_1 = \frac{1 + \sqrt{1 - 2\lambda}}{\sqrt{2}}, \qquad \alpha_2 = \frac{1 - \sqrt{1 - 2\lambda}}{\sqrt{2}} \tag{2.44}$$

Appendix B:

Consider two bases $\{|z,z\rangle,|z,-z\rangle,|-z,-z\rangle\}$ and $\{|b_1,b_2\rangle,|b'_1,b'_2\rangle,\|b"_1,b"_2\rangle\rangle\}$. Then imposing the MUBs condition (2.42) to each state of the first basis with $|b_1,b_2\rangle$ of the second basis leads to:

(I) $\left|\langle z,z|b_1,b_2\rangle\right|^2 = 1/3$

MUBs condition (2.42) gives

$$6\left(b_{1z} + b_{2z}\right) + 6\, b_{1z}b_{2z} = 2(b_1 \cdot b_2) \qquad\qquad (2.45)$$

(II) $\left|\langle z,-z|b_1,b_2\rangle\right|^2 = 1/3$

MUBs condition (2.42) gives

$$-6\, b_{1z}b_{2z} = -2(b_1 \cdot b_2) \qquad\qquad (2.46)$$

(III) Let $\left|\langle -z,-z|b_1,b_2\rangle\right|^2 = 1/3$

MUBs condition (2.42) gives

$$-6\left(b_{1z} + b_{2z}\right) + 6\, b_{1z}b_{2z} = 2(b_1 \cdot b_2) \qquad\qquad (2.47)$$

(2.45) + (2.46) $\Rightarrow\ b_{1z} + b_{2z} = 0\ or\ b_{1z} = -b_{2z}$ $\qquad\qquad (2.48)$

(2.45) + (2.47) $\Rightarrow 12 b_{1z}b_{2z} = 4 b_1 \cdot b_2\ or\ b_{1x}b_{2x} + b_{1y}b_{2y} = 2b_{1z}b_{2z}$ $\qquad\qquad (2.49)$

from (2.48) and (2.49)

$$b_{1x}b_{2x} + b_{1y}b_{2y} = -2b_{1z}^2 \qquad\qquad (2.50)$$

if $b_1 = \left(\sin\theta, 0, \cos\theta\right)$ and $b_2 = \left(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta\right)$ then eqn.(2.50) implies

$$\sin^2\theta\cos\varphi = -2\cos^2\theta \quad \text{or} \quad \cos\varphi = -2\cot^2\theta \tag{2.51}$$

To show $\left|a_1,a_2\right\rangle, \left|a_1',a_2'\right\rangle, \left|a_1'',a_2''\right\rangle$ **and** $\left|b_1,b_2\right\rangle, \left|b_1',b_2'\right\rangle, \left|b_1'',b_2''\right\rangle$ **are MUBs:**

Let

$$\vec{a}_1 = \left(x_1, y_1, z\right), \qquad \vec{a}_2 = \left(x_2, y_2, -z\right)$$

$$\vec{a}_1' = \left(\frac{-x_1 - \sqrt{3}y_1}{2}, \frac{\sqrt{3}x_1 - y_1}{2}, z\right), \quad \vec{a}_2' = \left(\frac{-x_2 - \sqrt{3}y_2}{2}, \frac{\sqrt{3}x_2 - y_2}{2}, -z\right)$$

$$\vec{a}_1'' = \left(\frac{-x_1 + \sqrt{3}y_1}{2}, \frac{-\sqrt{3}x_1 - y_1}{2}, z\right), \quad \vec{a}_2'' = \left(\frac{-x_2 + \sqrt{3}y_2}{2}, \frac{-\sqrt{3}x_2 - y_2}{2}, -z\right)$$

and

$$\vec{b}_1 = \left(u_1, v_1, w\right), \qquad \vec{b}_2 = \left(u_2, v_2, -w\right)$$

$$\vec{b}_1' = \left(\frac{-u_1 - \sqrt{3}v_1}{2}, \frac{\sqrt{3}u_1 - v_1}{2}, w\right), \quad \vec{b}_2' = \left(\frac{-u_2 - \sqrt{3}v_2}{2}, \frac{\sqrt{3}u_2 - v_2}{2}, -w\right)$$

$$\vec{b}_1'' = \left(\frac{-u_1 + \sqrt{3}v_1}{2}, \frac{-\sqrt{3}u_1 - v_1}{2}, w\right), \quad \vec{b}_2'' = \left(\frac{-u_2 + \sqrt{3}v_2}{2}, \frac{-\sqrt{3}u_2 - v_2}{2}, -w\right)$$

Consider the case:

$$z = 0, \quad x_1 = 1, \quad y_1 = 0, \quad x_2 = 0, \quad y_2 = 1$$

$$u_1 = \sin\theta\cos\varphi_0, \quad v_1 = \sin\theta\sin\varphi_0$$

$$w = \cos\theta$$

$$u_2 = \sin\theta\cos\left(\varphi + \varphi_0\right), \quad v_2 = \sin\theta\sin\left(\varphi + \varphi_0\right)$$

Call LHS of (2.44) F_{11} for $|a_1, a_2\rangle$ and $|b_1, b_2\rangle$

$$F_{11} = (a_1 + a_2) \cdot (b_1 + b_2) + (a_1 \cdot b_1)(a_2 \cdot b_2) + (a_1 \cdot b_2)(a_2 \cdot b_1)$$
$$= u_1 + u_2 + v_1 + v_2 + u_1 v_2 + u_2 v_1$$

$$F_{11} = 2\sin\theta \left[\cos\frac{\varphi}{2} \left\{ \cos\left(\frac{\varphi}{2} + \varphi_0\right) + \sin\left(\frac{\varphi}{2} + \varphi_0\right) \right\} + \sin\theta \sin\left(\frac{\varphi}{2} + \varphi_0\right) \cos\left(\frac{\varphi}{2} + \varphi_0\right) \right] \qquad (2.52)$$

$$F_{11} = 0 \Rightarrow \sin\theta = -\cos\frac{\varphi}{2} \left(\frac{1}{\sin\left(\varphi_0 + \frac{\varphi}{2}\right)} + \frac{1}{\cos\left(\varphi_0 + \frac{\varphi}{2}\right)} \right) \qquad (2.53)$$

Call LHS of (2.44) F_{21} for $|a'_1, a'_2\rangle$ and $|b_1, b_2\rangle$

$$F_{21} = (a'_1 + a'_2) \cdot (b_1 + b_2) + (a'_1 \cdot b_1)(a'_2 \cdot b_2) + (a'_1 \cdot b_2)(a'_2 \cdot b_1)$$

$$F_{21} = -\frac{1}{2} F_{11} - \frac{\sqrt{3}}{2} \left[u_1 + u_2 - v_1 - v_2 - u_1 u_2 + v_1 v_2 \right]$$

$$F_{21} = -\frac{1}{2} F_{11} - \frac{\sqrt{3}}{2} \left[\sin\theta \left\{ 2\cos\frac{\varphi}{2}(\cos\alpha - \sin\alpha) \right\} - \sin^2\theta \cos 2\alpha \right]$$

where $\alpha = \varphi_0 + \dfrac{\varphi}{2}$ $\qquad (2.54)$

$$F_{21} = 0 \ \textit{if} \ F_{11} = 0 \ \text{and} \ \sin\theta = \frac{2\cos(\varphi/2)(\cos\alpha - \sin\alpha)}{2\cos\alpha} = \frac{2\cos(\varphi/2)}{\cos\alpha + \sin\alpha} \qquad (2.55)$$

Call LHS of (2.44) F_{31} for $|a''_1, a''_2\rangle$ and $|b_1, b_2\rangle$

$$F_{31} = (a''_1 + a''_2) \cdot (b_1 + b_2) + (a''_1 \cdot b_1)(a''_2 \cdot b_2) + (a''_1 \cdot b_2)(a''_2 \cdot b_1)$$

$$F_{31} = -\frac{1}{2}F_{11} + \frac{\sqrt{3}}{2}\left[u_1 + u_2 - v_1 - v_2 - u_1 u_2 + v_1 v_2\right]$$

$$F_{31} = -\frac{1}{2}F_{11} + \frac{\sqrt{3}}{2}\left[\sin\theta\left\{2\cos\frac{\varphi}{2}(\cos\alpha - \sin\alpha)\right\} - \sin^2\theta\cos 2\alpha\right]$$

$F_{31} = 0$ if $F_{11} = 0$ and (2.48), as above.

Therefore in order for $|a_1, a_2\rangle$, $|a'_1, a'_2\rangle$ and $|a''_1, a''_2\rangle$ to be MUB to $|b_1, b_2\rangle$ equation (2.51) and (2.53) must be satisfied.

Comparing (2.53) and (2.55) we get

$$\frac{2\cos(\varphi/2)}{\cos\alpha + \sin\alpha} = -\cos\frac{\varphi}{2}\left(\frac{1}{\sin\alpha} + \frac{1}{\cos\alpha}\right)$$

$$\Rightarrow \sin 2\alpha = -\frac{1}{2} \tag{2.56}$$

Thus

$$2\alpha = 2\varphi_0 + \varphi = \pi + \frac{\pi}{6} \ or \ 2\pi - \frac{\pi}{6}$$

$$\Rightarrow \varphi_0 = \frac{7\pi}{12} - \varphi \ or \ \frac{11\pi}{12} - \varphi$$

From (2.53) and (2.54) we have

$$\sin\theta = -\cos\frac{\varphi}{2}\left(\frac{1}{\sin\alpha} + \frac{1}{\cos\alpha}\right)$$

But $\quad \alpha = \dfrac{\pi}{2} + \dfrac{\pi}{12} \ or \ \pi - \dfrac{\pi}{12}$

Case 1: when $\alpha = \dfrac{\pi}{2} + \dfrac{\pi}{12}$

$$\sin\theta = -\cos\frac{\varphi}{2}\left(\frac{1}{\cos(\pi/12)} - \frac{1}{\sin(\pi/12)}\right)$$

Case 2: when $\alpha = \pi - \dfrac{\pi}{12}$

$$\sin\theta = \cos\frac{\varphi}{2}\left(\frac{1}{\cos(\pi/12)} - \frac{1}{\sin(\pi/12)}\right)$$

Define

$$\Gamma \equiv \left(\frac{1}{\cos(\pi/12)} - \frac{1}{\sin(\pi/12}\right)^2$$

Either of the case 1 or 2 when squared gives

$$\sin^2\theta = \cos^2\frac{\varphi}{2}\Gamma$$

From (2.49)

$$1 - \cos^2\theta = \frac{1 - 2\cot^2\theta}{2}\Gamma$$

Let $x = \cos^2\theta$, $\cot^2\theta = \dfrac{\cos^2\theta}{1-\cos^2\theta} = \dfrac{x}{1-x}$

Above expression reduces to a quadratic equation in x:

$$2x^2 + x(3\Gamma - 4) + 2 - \Gamma = 0$$

$$x = \frac{(4 - 3\Gamma) \pm \sqrt{9\Gamma^2 - 16\Gamma}}{4}$$

$$\Gamma \equiv \left(\frac{1}{\cos(\pi/12)} - \frac{1}{\sin(\pi/12)} \right)^2 = \left(\frac{\sin(\pi/12) - \cos(\pi/12)}{\cos(\pi/12)\sin(\pi/12)} \right)^2 = 4 \frac{(1 - \sin(\pi/6))}{\sin^2(\pi/6)} = 8$$

Substituting Γ into x roots:

$$x = -5 \pm 2\sqrt{7}$$

or

$$\cos\theta = \sqrt{-5 \pm 2\sqrt{7}}$$
$$\Rightarrow \theta = 57.3225°$$

Similarly

$$F_{12} = (a_1 + a_2) \cdot (b_1' + b_2') + (a_1 \cdot b_1')(a_2 \cdot b_2') + (a_1 \cdot b_2')(a_2 \cdot b_1')$$
$$= -\frac{1}{2} F_{11} + \frac{\sqrt{3}}{2} \left[u_1 + u_2 - v_1 - v_2 - u_1 u_2 + v_1 v_2 \right]$$

$F_{12} = 0$ *if* $F_{11} = 0$ and (2.55) is satisfied.

$$F_{13} = (a_1 + a_2) \cdot (b_1'' + b_2'') + (a_1 \cdot b_1'')(a_2 \cdot b_2'') + (a_1 \cdot b''_2)(a_2 \cdot b_1'')$$
$$= -\frac{1}{2} F_{11} - \frac{\sqrt{3}}{2} \left[u_1 + u_2 - v_1 - v_2 - u_1 u_2 + v_1 v_2 \right]$$

$F_{13} = 0$ *if* $F_{11} = 0$ and (2.55) is satisfied.

$$F_{22} = (a'_1 + a'_2) \cdot (b_1' + b_2') + (a'_1 \cdot b_1')(a'_2 \cdot b_2') + (a'_1 \cdot b'_2)(a'_2 \cdot b_1')$$

$$= -\frac{1}{2}F_{11} + \frac{\sqrt{3}}{2}\left[u_1 + u_2 - v_1 - v_2 - u_1 u_2 + v_1 v_2\right]$$

$F_{22} = 0$ *if* $F_{11} = 0$ and (2.55) is satisfied.

$$F_{23} = (a'_1 + a'_2) \cdot (b_1'' + b_2'') + (a'_1 \cdot b_1'')(a'_2 \cdot b_2'') + (a'_1 \cdot b''_2)(a'_2 \cdot b_1'')$$

$$= -\frac{1}{2}F_{11} + \frac{\sqrt{3}}{2}\left[u_1 + u_2 - v_1 - v_2 - u_1 u_2 + v_1 v_2\right]$$

$F_{23} = 0$ *if* $F_{11} = 0$ and (2.55) is satisfied.

$$F_{32} = (a''_1 + a''_2) \cdot (b_1' + b_2') + (a''_1 \cdot b_1')(a''_2 \cdot b_2') + (a''_1 \cdot b'_2)(a''_2 \cdot b_1')$$

$$= -\frac{1}{2}F_{11} + \frac{\sqrt{3}}{2}\left[u_1 + u_2 - v_1 - v_2 - u_1 u_2 + v_1 v_2\right]$$

$F_{32} = 0$ *if* $F_{11} = 0$, and (2.55) is satisfied.

$$F_{32} = (a''_1 + a''_2) \cdot (b_1'' + b_2'') + (a''_1 \cdot b_1'')(a''_2 \cdot b_2'') + (a''_1 \cdot b'_2)(a''_2 \cdot b_1'')$$

$$= -\frac{1}{2}F_{11} + \frac{\sqrt{3}}{2}\left[u_1 + u_2 - v_1 - v_2 - u_1 u_2 + v_1 v_2\right]$$

$F_{32} = 0$ *if* $F_{11} = 0$, and (2.55) is satisfied.

All of these bases satisfy the same condition of MUBs. Therefore
$\left|a_1, a_2\right\rangle, \left|a_1', a_2'\right\rangle, \left|a_1'', a_2''\right\rangle$ and $\left|b_1, b_2\right\rangle, \left|b_1', b_2'\right\rangle, \left|b_1'', b_2''\right\rangle$ are MUBs.

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